

DETAILED DERIVATIONS FOR MINQUE AND MIVQUE ESTIMATION
OF VARIANCE COMPONENTS FROM UNBALANCED DATA
IN THE 1-WAY CLASSIFICATION RANDOM MODEL

BU-448-M

February 1973

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ABSTRACT

This paper provides detailed derivation of MINQUE and MIVQUE estimators of variance components from unbalanced data in the 1-way classification random model, assuming normality and homoscedasticity of error terms. Two forms of the model are considered: where $\mu \equiv 0$ and where $\mu \neq 0$. The variances and covariances of the estimators are also derived. It is shown that for balanced data these estimators and their variances and covariances reduce to the familiar expressions for the usual AOV estimators in such cases. More general discussion of MINQUE, MIVQUE, and the 1-way classification models being considered here is given in BU-447-M.

APPENDIX

I. Detailed Derivations for MINQUE and MIVQUE Estimation of Variance Components in the Unbalanced 1-way Classification Model

1. MIVQUE for the model with $\mu \equiv 0$

The model may be written as

$$y = \left(\sum_i^+ 1_{n_i} \right) \underline{a} + I_N \underline{e} = Z_1 \underline{a} + Z_2 \underline{e} \quad .$$

$$\text{Then } V_1 = Z_1 Z_1' = \sum_i^+ J_{n_i} \quad ,$$

$$V_2 = Z_2 Z_2' = I_N \quad ,$$

$$\text{and } V = \sigma_a^2 V_1 + \sigma_e^2 V_2 = \sum_i^+ \left(\sigma_e^2 I_{n_i} + \sigma_a^2 J_{n_i} \right) \quad . \quad (A1)$$

Under the assumption that $\mu \equiv 0$, one operates with Rao's general expressions for MINQUE and MIVQUE taking $X = 0$ and therefore

$$P_1 = X(X'V^{-1}X)^{-1}X'V^{-1} = 0 \quad .$$

$$\text{Then } R_1 = V^{-1}(I_N - P_1) = V^{-1}$$

$$= \sum_i^+ \left[\frac{1}{\sigma_e^2} I_{n_i} - \frac{\sigma_a^2}{\sigma_e^2(\sigma_e^2 + n_i \sigma_a^2)} J_{n_i} \right] \quad . \quad (A2)$$

We need to derive $s_{1:11} = \text{tr}(V_1 R_1)^2$, $s_{1:12} = \text{tr}(V_2 R_1 V_1 R_1)$, $s_{1:22} = \text{tr}(V_2 R_1)^2$, $u_{1:1} = y' R_1 V_1 R_1 y$, and $u_{1:2} = y' R_1 V_2 R_1 y$.

From (A1) and (A2)

$$\begin{aligned} V_1 R_1 &= \sum^+ \left[\frac{1}{\sigma_e^2} J_{n_1} - \frac{n_i \sigma_a^2}{\sigma_e^2 (\sigma_e^2 + n_i \sigma_a^2)} J_{n_1} \right] \\ &= \sum^+ \frac{1}{\sigma_e^2 + n_i \sigma_a^2} J_{n_1} ; \end{aligned}$$

$$\begin{aligned} R_1 V_1 R_1 &= \sum^+ \left[\frac{1}{\sigma_e^2 (\sigma_e^2 + n_i \sigma_a^2)} J_{n_1} - \frac{n_i \sigma_a^2}{\sigma_e^2 (\sigma_e^2 + n_i \sigma_a^2)^2} J_{n_1} \right] \\ &= \sum^+ \frac{1}{(\sigma_e^2 + n_i \sigma_a^2)^2} J_{n_1} ; \end{aligned} \quad (A3)$$

$$(V_1 R_1)^2 = \sum^+ \frac{n_i}{(\sigma_e^2 + n_i \sigma_a^2)^2} J_{n_1} ; \quad (A4)$$

$$V_2 R_1 = R_1 = V^{-1} \text{ as given in (A2) ;}$$

$$\begin{aligned} R_1 V_2 R_1 &= (V_2 R_1)^2 = R_1^2 \\ &= \sum^+ \left[\frac{1}{\sigma_e^4} I_{n_1} + \left(\frac{-2\sigma_a^2}{\sigma_e^4 (\sigma_e^2 + n_i \sigma_a^2)} + \frac{n_i \sigma_a^4}{\sigma_e^4 (\sigma_e^2 + n_i \sigma_a^2)^2} \right) J_{n_1} \right] \\ &= \sum^+ \left[\frac{1}{\sigma_e^4} I_{n_1} - \frac{2\sigma_e^2 \sigma_a^2 + n_i \sigma_a^4}{\sigma_e^4 (\sigma_e^2 + n_i \sigma_a^2)^2} J_{n_1} \right] ; \end{aligned} \quad (A5)$$

$$\text{and } R_1 V_1 R_1 V_2 = R_1 V_1 R_1 \text{ of (A3) .} \quad (A6)$$

In what follows it is convenient to let

$$q_1 = \frac{1}{1 + n_i \frac{\sigma_a^2}{\sigma_e^2}} = \frac{\sigma_e^2}{\sigma_e^2 + n_i \sigma_a^2} . \quad (A7)$$

This notation duplicates that of Townsend and Searle (1971) and facilitates showing the equivalence of their results and those obtained here.

Using (A4), (A5), and (A6) gives

$$s_{1:11} = \text{tr}(V_1 R_1)^2 = \sum \frac{n_i^2}{(\sigma_e^2 + n_i \sigma_a^2)^2} = \frac{1}{\sigma_e^4} \sum n_i^2 q_i^2, \quad (\text{A8})$$

$$s_{1:12} = \text{tr}(V_1 R_1 V_2 R_1) = \sum \frac{n_i}{(\sigma_e^2 + n_i \sigma_a^2)^2} = \frac{1}{\sigma_e^4} \sum n_i q_i^2, \quad (\text{A9})$$

$$\begin{aligned} \text{and } s_{1:22} &= \text{tr}(V_2 R_1)^2 = \sum \left[\frac{n_i \sigma_e^4 + 2n_i(n_i - 1)\sigma_e^2 \sigma_a^2 + n_i^2(n_i - 1)\sigma_a^4}{\sigma_e^4(\sigma_e^2 + n_i \sigma_a^2)^2} \right] \\ &= \frac{1}{\sigma_e^4} \sum q_i^2 \left[n_i + 2n_i(n_i - 1) \frac{\sigma_a^2}{\sigma_e^2} + n_i^2(n_i - 1) \frac{\sigma_a^4}{\sigma_e^4} \right] \\ &= \frac{1}{\sigma_e^4} \sum q_i^2 \left[(n_i - 1) \left(1 + n_i \frac{\sigma_a^2}{\sigma_e^2} \right)^2 + 1 \right] \\ &= \frac{1}{\sigma_e^4} \sum q_i^2 \left[(n_i - 1) q_i^{-2} + 1 \right] \\ &= \frac{1}{\sigma_e^4} \left[\sum q_i^2 + N - a \right]. \end{aligned} \quad (\text{A10})$$

From (A3) and (A5)

$$u_{1:1} = y' R_1 V_1 R_1 y = \sum_i \frac{y_i^2}{(\sigma_e^2 + n_i \sigma_a^2)^2} = \frac{1}{\sigma_e^4} \sum_i q_i^2 y_i^2. \quad (\text{A11})$$

$$\text{and } u_{1:2} = y' R_1 V_2 R_1 y$$

$$\begin{aligned}
 &= \sum_i \sum_j \frac{y_{ij}^2}{\sigma_e^4} - \sum_i \frac{2\sigma_e^2 \sigma_a^2 + n_i \sigma_a^4}{\sigma_e^4 (\sigma_e^2 + n_i \sigma_a^2)^2} y_{i.}^2 \\
 &= \frac{1}{\sigma_e^4} \left[\sum_i \sum_j y_{ij}^2 - \sum_i \left\{ 2 \frac{\sigma_a^2}{\sigma_e^2} q_i^2 + n_i \frac{\sigma_a^4}{\sigma_e^4} q_i^2 \right\} y_{i.}^2 \right] \\
 &= \frac{1}{\sigma_e^4} \left[\sum_i \sum_j y_{ij}^2 - \sum_i q_i^2 \left\{ \frac{1}{n_i} \left(1 + n_i \frac{\sigma_a^2}{\sigma_e^2} \right)^2 - \frac{1}{n_i} \right\} y_{i.}^2 \right] \\
 &= \frac{1}{\sigma_e^4} \left[\sum_i \sum_j y_{ij}^2 - \sum_i q_i^2 \left\{ \frac{1}{n_i} q_i^{-2} - \frac{1}{n_i} \right\} y_{i.}^2 \right] \\
 &= \frac{1}{\sigma_e^4} \left[\sum_i \sum_j y_{ij}^2 - \sum_i \frac{y_{i.}^2}{n_i} + \sum_i \frac{q_i^2 y_{i.}^2}{n_i} \right] \quad (A12)
 \end{aligned}$$

Then the estimators of σ_e^2 and σ_a^2 are

$$\hat{\sigma}_{le}^2 = \frac{1}{\Delta_1} \left[-s_{1:12} u_{1:1} + s_{1:11} u_{1:2} \right] \quad (A13)$$

$$\text{and} \quad \hat{\sigma}_{la}^2 = \frac{1}{\Delta_1} \left[s_{1:22} u_{1:1} - s_{1:12} u_{1:2} \right] \quad (A14)$$

using (A8) through (A12) and letting $\Delta_1 = s_{1:11} s_{1:22} - s_{1:12}^2$ = the determinant of S_1 . Equations (A13) and (A14) may be written in terms of (A8) through (A12) as

$$\hat{\sigma}_{le}^2 = \frac{-\left[\frac{1}{\sigma_e^4} \sum n_i q_i^2 \right] \left[\frac{1}{\sigma_e^4} \sum q_i^2 y_{i.}^2 \right] + \left[\frac{1}{\sigma_e^4} \sum n_i q_i^2 \right] \left[\frac{1}{\sigma_e^4} \left(\sum_i \sum_j y_{ij}^2 - \sum_i \frac{y_{i.}^2}{n_i} + \sum_i \frac{q_i^2 y_{i.}^2}{n_i} \right) \right]}{\left[\frac{1}{\sigma_e^4} \sum n_i^2 q_i^2 \right] \left[\frac{1}{\sigma_e^4} \left(\sum_i q_i^2 + N - a \right) \right] - \left[\frac{1}{\sigma_e^4} \sum n_i q_i^2 \right]^2}$$

$$= \frac{\sum \left[\left(\sum n_i^2 q_i^2 - n_i \sum n_i q_i^2 \right) \frac{q_i^2 y_i^2}{n_i} + \sum n_i^2 q_i^2 \left[\sum \sum y_{ij}^2 - \sum \frac{y_i^2}{n_i} \right] \right]}{\left[\sum n_i^2 q_i^2 \right] \left[\sum q_i^2 + N - a \right] - \left[\sum n_i q_i^2 \right]^2} \quad (A15)$$

$$\text{and } \hat{\sigma}_{1a}^2 = \frac{\frac{1}{\sigma_e^4} \left[\sum q_i^2 + N - a \right] \left[\frac{1}{\sigma_e^4} \sum q_i^2 y_i^2 \right] - \left[\frac{1}{\sigma_e^4} \sum n_i q_i^2 \right] \left[\frac{1}{\sigma_e^4} \left(\sum \sum y_{ij}^2 - \sum \frac{y_i^2}{n_i} + \sum \frac{q_i^2 y_i^2}{n_i} \right) \right]}{\left[\frac{1}{\sigma_e^4} \sum n_i^2 q_i^2 \right] \left[\frac{1}{\sigma_e^4} \left(\sum q_i^2 + N - a \right) \right] - \left[\frac{1}{\sigma_e^4} \sum n_i q_i^2 \right]^2}$$

$$= \frac{\sum \left[n_i \left(\sum q_i^2 + N - a \right) - \sum n_i q_i^2 \right] \frac{q_i^2 y_i^2}{n_i} - \sum n_i q_i^2 \left[\sum \sum y_{ij}^2 - \sum \frac{y_i^2}{n_i} \right]}{\left[\sum n_i^2 q_i^2 \right] \left[\sum q_i^2 + N - a \right] - \left[\sum n_i q_i^2 \right]^2} \quad (A16)$$

Equations (A15) and (A16) are identical respectively to equations (2) and (4) of Townsend and Searle (1971) for these estimators.

Using the results that $v(y'Ay) = 2 \text{ tr}(VA)^2$ and $\text{cov}(y'Ay, y'By) = 2 \text{ tr}(VAVB)$ when $y \sim N(0, V)$, the variances and covariance of the estimators can be written as

$$v(\hat{\sigma}_{1e}^2) = \frac{1}{\Delta_1^2} \left[s_{1:12}^2 v(u_{1:1}) + s_{1:11}^2 v(u_{1:2}) - 2s_{1:11}s_{1:12} \text{cov}(u_{1:1}, u_{1:2}) \right], \quad (A17)$$

$$v(\hat{\sigma}_{1a}^2) = \frac{1}{\Delta_1^2} \left[s_{1:22}^2 v(u_{1:1}) + s_{1:12}^2 v(u_{1:2}) - 2s_{1:12}s_{1:22} \text{cov}(u_{1:1}, u_{1:2}) \right], \quad (A18)$$

$$\text{and } \text{cov}(\hat{\sigma}_{1e}^2, \hat{\sigma}_{1a}^2) = \frac{1}{\Delta_1^2} \left[-s_{1:12}s_{1:22} v(u_{1:1}) - s_{1:11}s_{1:12} v(u_{1:2}) + (s_{1:11}s_{1:22} + s_{1:12}^2) \text{cov}(u_{1:1}, u_{1:2}) \right], \quad (A19)$$

leaving only $v(u_{1:1})$, $v(u_{1:2})$, and $\text{cov}(u_{1:1}, u_{1:2})$ to be derived. Noting that $R_1 = V^{-1}$,

$$v(u_{1:1}) = 2\text{tr}(VR_1 V_1 R_1)^2 = 2\text{tr}(V_1 R_1)^2 = 2s_{1:11} \quad , \quad (\text{A20})$$

$$v(u_{1:2}) = 2\text{tr}(VR_1 V_2 R_1)^2 = 2\text{tr}(V_2 R_1)^2 = 2s_{1:22} \quad , \quad (\text{A21})$$

$$\text{and } \text{cov}(u_{1:1}, u_{1:2}) = 2\text{tr}(VR_1 V_1 R_1 VR_1 V_2 R_1) = 2\text{tr}(V_1 R_1 V_2 R_1) = 2s_{1:12} \quad , \quad (\text{A22})$$

these results coming from (A8), (A9), and (A10).

Substituting (A20), (A21), and (A22) into (A17), (A18), and (A19) yields

$$\begin{aligned} v(\hat{\sigma}_{1e}^2) &= \frac{2}{\Delta_1^2} [s_{1:12}^2 s_{1:11} + s_{1:11}^2 s_{1:22} - 2s_{1:11} s_{1:12}^2] \\ &= \frac{2s_{1:11} \Delta_1}{\Delta_1^2} = \frac{2s_{1:11}}{\Delta_1} \end{aligned} \quad (\text{A23})$$

and similarly

$$v(\hat{\sigma}_{1a}^2) = \frac{2s_{1:22}}{\Delta_1} \quad (\text{A24})$$

$$\text{and } \text{cov}(\hat{\sigma}_{1e}^2, \hat{\sigma}_{1a}^2) = \frac{-2s_{1:12}}{\Delta_1} \quad (\text{A25})$$

Substituting (A8), (A9), and (A10) into (A23) and (A24) immediately displays the forms for $v(\hat{\sigma}_{1e}^2)$ and $v(\hat{\sigma}_{1a}^2)$ given in equations (3) and (5) of Townsend and Searle (1971).

2. MINQUE for the model with $\mu \equiv 0$

The minimization for this case differs from MINQUE only in the substitution of

$$V_u = V_1 + V_2 \quad (\text{A26})$$

for $V = \sigma_a^2 V_1 + \sigma_e^2 V_2$ in (A2). Therefore, although the estimators could be derived by a series of steps completely analogous to those used for MIVQUE, it is simpler to obtain the estimators as a special case of MIVQUE, setting $\sigma_e^2 = \sigma_a^2 = 1$ in equations (A2) through (A14). Doing this in (A7) we then re-write q_i as w_i where

$$w_i = \frac{1}{1 + n_i} \quad . \quad (A27)$$

From (A8) through (A12) we get

$$s_{2:11} = \sum \frac{n_i^2}{(1 + n_i)^2} = \sum n_i^2 w_i^2 \quad , \quad (A28)$$

$$s_{2:12} = \sum \frac{n_i}{(1 + n_i)^2} = \sum n_i w_i^2 \quad , \quad (A29)$$

$$\begin{aligned} s_{2:22} &= \sum \frac{1}{(1 + n_i)^2} + N - a \\ &= \sum w_i^2 + N - a \quad , \end{aligned} \quad (A30)$$

$$u_{2:1} = \sum \frac{y_i^2}{(1 + n_i)^2} = \sum w_i^2 y_i^2 \quad , \quad (A31)$$

and

$$\begin{aligned} u_{2:2} &= \sum \sum y_{ij}^2 - \sum \frac{y_{i.}^2}{n_i} + \sum \frac{y_{i.}^2}{n_i(1 + n_i)^2} \\ &= \sum \sum y_{ij}^2 - \sum \frac{y_{i.}^2}{n_i} + \sum \frac{w_i^2 y_{i.}^2}{n_i} \quad . \end{aligned} \quad (A32)$$

Letting $\Delta_2 = s_{2:11}s_{2:22} - s_{2:12}^2$ = the determinant of S_2 and using (A28) through (A32), the estimators may be written as

$$\hat{\sigma}_{2e}^2 = \frac{1}{\Delta_2} \left[-s_{2:12}u_{2:1} + s_{2:11}u_{2:2} \right] \quad (A33)$$

and
$$\hat{\sigma}_{2a}^2 = \frac{1}{\Delta_e^2} \left[s_{2:22} u_{2:1} - s_{2:12} u_{2:2} \right] . \quad (A34)$$

The variances and covariance of the estimators must be derived directly. They take the form

$$v(\hat{\sigma}_{2e}^2) = \frac{1}{\Delta_e^2} \left[s_{2:12}^2 v(u_{2:1}) + s_{2:11}^2 v(u_{2:2}) - 2s_{2:11}s_{2:12} \text{cov}(u_{2:1}, u_{2:2}) \right] , \quad (A35)$$

$$v(\hat{\sigma}_{2a}^2) = \frac{1}{\Delta_e^2} \left[s_{2:22}^2 v(u_{2:1}) + s_{2:12}^2 v(u_{2:2}) - 2s_{2:12}s_{2:22} \text{cov}(u_{2:1}, u_{2:2}) \right] , \quad (A36)$$

and
$$\text{cov}(\hat{\sigma}_{2e}^2, \hat{\sigma}_{2a}^2) = \frac{1}{\Delta_e^2} \left[-s_{2:12}s_{2:22} v(u_{2:1}) - s_{2:11}s_{2:12} v(u_{2:2}) + (s_{2:11}s_{2:22} + s_{2:12}^2) \text{cov}(u_{2:1}, u_{2:2}) \right] , \quad (A37)$$

where $v(u_{2:1})$, $v(u_{2:2})$, and $\text{cov}(u_{2:1}, u_{2:2})$ are given below in (A39), (A41), and (A42).

In the following we will use

$$R_2 V_1 R_2 = \sum^+ w_i^2 J_{n_i}$$

and
$$R_2 V_2 R_2 = \sum^+ \left[I_{n_i} - w_i^2 (n_i + 2) J_{n_i} \right] ,$$

obtained from (A3) and (A5) by setting $\sigma_e^2 = \sigma_a^2 = 1$, and q_i as defined in (A7).

$$v(u_{2:1}) = 2\text{tr}(VR_2 V_1 R_2)^2 \quad \text{where}$$

$$\begin{aligned} VR_2 V_1 R_2 &= \sum^+ \left[w_i^2 \sigma_e^2 + w_i^2 n_i \sigma_a^2 \right] J_{n_i} \\ &= \sum^+ \frac{w_i^2 \sigma_e^2}{q_i} J_{n_i} \end{aligned} \quad (A38)$$

and
$$(VR_2 V_1 R_2)^2 = \sum + \frac{n_i w_i^4 \sigma_e^4}{q_i^2} J_{n_i} .$$

Then
$$v(u_{2:1}) = 2\sigma_e^4 \sum \frac{n_i^2 w_i^4}{q_i^2} . \quad (A39)$$

Similarly,

$$v(u_{2:2}) = 2\text{tr}(VR_2 V_2 R_2)^2 \quad \text{for}$$

$$\begin{aligned} VR_2 V_2 R_2 &= \sum + \left[\sigma_e^2 I_{n_i} + \left(\sigma_a^2 - w_i^2 [n_i + 2] \sigma_e^2 - n_i w_i^2 [n_i + 2] \sigma_a^2 \right) J_{n_i} \right] \\ &= \sum + \left[\sigma_e^2 I_{n_i} + w_i^2 \left(\sigma_a^2 - [n_i + 2] \sigma_e^2 \right) J_{n_i} \right] \end{aligned} \quad (A40)$$

$$\begin{aligned} \text{and } (VR_2 V_2 R_2)^2 &= \sum + \left[\sigma_e^4 I_{n_i} + \left\{ 2w_i^2 \sigma_e^2 \left(\sigma_a^2 - [n_i + 2] \sigma_e^2 \right) \right. \right. \\ &\quad \left. \left. + n_i w_i^4 \left(\sigma_a^2 - [n_i + 2] \sigma_e^2 \right)^2 \right\} J_{n_i} \right] \\ &= \sum + \left[\sigma_e^4 I_{n_i} + w_i^4 \left(-n_i^3 \sigma_e^4 - 4n_i^2 \sigma_e^4 - 6n_i \sigma_e^4 \right. \right. \\ &\quad \left. \left. + 2\sigma_e^2 \sigma_a^2 - 4\sigma_e^4 + n_i \sigma_a^4 \right) J_{n_i} \right] \\ &= \sum + \left[\sigma_e^4 I_{n_i} + w_i^4 \left(\frac{-w_i^4 \sigma_e^4 + (\sigma_e^2 + n_i \sigma_a^2)^2}{n_i} \right) J_{n_i} \right] \\ &= \sum + \left[\sigma_e^4 I_{n_i} - \left(\frac{\sigma_e^4}{n_i} - \frac{w_i^4 \sigma_e^4}{n_i q_i^2} \right) J_{n_i} \right] . \end{aligned}$$

So
$$\begin{aligned} v(u_{2:2}) &= 2 \left[N \sigma_e^4 - a \sigma_e^4 + \sum \frac{w_i^4 \sigma_e^4}{q_i^2} \right] \\ &= 2\sigma_e^4 \left[\sum \frac{w_i^4}{q_i^2} + N - a \right] . \end{aligned} \quad (A41)$$

$$\text{cov}(u_{2:1}, u_{2:2}) = 2\text{tr}(VR_2 V_1 R_2 VR_2 V_2 R_2) \text{ where from (A38)}$$

and (A40)

$$\begin{aligned} VR_2 V_1 R_2 VR_2 V_2 R_2 &= \sum^+ \left[\frac{w_i^2 \sigma_e^4}{q_i} + \frac{n_i w_i^4 \sigma_e^2}{q_i} (\sigma_a^2 - [n_i + 2] \sigma_e^2) \right] J_{n_i} \\ &= \sum^+ \frac{w_i^4 \sigma_e^2 (\sigma_e^2 + n_i \sigma_a^2)}{q_i} J_{n_i} \\ &= \sum^+ \frac{w_i^4 \sigma_e^4}{q_i^2} J_{n_i} . \end{aligned}$$

Thus

$$\text{cov}(u_{2:1}, u_{2:2}) = 2\sigma_e^4 \sum \frac{n_i w_i^4}{q_i^2} . \quad (\text{A42})$$

3. MIVQUE for the model with $\mu \neq 0$

Writing the model as

$$y = \mu 1_N + \left(\sum^+ 1_{n_i} \right) \underline{a} + I_N \underline{e} = X\mu + Z_1 \underline{a} + Z_2 \underline{e}$$

implies

$$V_1 = Z_1 Z_1' = \sum^+ J_{n_i} ,$$

$$V_2 = Z_2 Z_2' = I_N ,$$

and

$$V = \sigma_a^2 V_1 + \sigma_e^2 V_2 = \sum^+ \left(\sigma_e^2 I_{n_i} + \sigma_a^2 J_{n_i} \right) .$$

Then

$$V^{-1} = \sum^+ \left[\frac{1}{\sigma_e^2} I_{n_i} - \frac{\sigma_a^2}{\sigma_e^2 (\sigma_e^2 + n_i \sigma_a^2)} J_{n_i} \right]$$

and
$$X'V^{-1}X = 1_N'V^{-1}1_N = \sum \frac{n_i}{\sigma_e^2 + n_i\sigma_a^2} .$$

We write

$$k_i = \frac{n_i}{\sigma_e^2 + n_i\sigma_a^2} \text{ and } k = \frac{1}{\sum k_i} \quad (\text{A43})$$

as notational shorthand, with $n_i q_i = k_i \sigma_e^2$,

$$X'V^{-1}X = \frac{1}{k}$$

and
$$(X'V^{-1}X)^{-1} = k .$$

Then

$$\begin{aligned} P_3 &= X(X'V^{-1}X)^{-1}X'V^{-1} \\ &= k1_N \left(\frac{1}{\sigma_e^2 + n_1\sigma_a^2} 1_{n_1}' \cdots \frac{1}{\sigma_e^2 + n_a\sigma_a^2} 1_{n_a}' \right) \\ &= [P_{3:1} P_{3:2} \cdots P_{3:a}] \text{ where } P_{3:i} = \frac{kk_i}{n_i} J_{N \times n_i} . \end{aligned}$$

$R_3 = V^{-1}(I - P_3)$ and its products with V_1 and V_2 have nonzero off-diagonal blocks, unlike the situation where $\mu \equiv 0$. It is convenient to partition these matrices as $M = \{M_{ii'}\}$ where $i, i' = 1, 2, \dots, a$, and to derive results by developing expressions for typical diagonal and off-diagonal blocks or sub-matrices.

$$R_3 = V^{-1}(I_N - P_3) \text{ has as its } i^{\text{th}} \text{ diagonal block}$$

$$[i^{\text{th}} \text{ diagonal block of } V^{-1}][i^{\text{th}} \text{ diagonal block of } (I_N - P_3)]$$

$$= \left[\frac{1}{\sigma_e^2} I_{n_i} - \frac{k_i \sigma_a^2}{n_i \sigma_e^2} J_{n_i} \right] \left[I_{n_i} - \frac{kk_i}{n_i} J_{n_i} \right]$$

$$\begin{aligned}
 &= \frac{1}{\sigma_e^2} I_{n_i} - \left[\frac{kk_i}{n_i \sigma_e^2} + \frac{k_i \sigma_a^2}{n_i \sigma_e^2} - \frac{kk_i^2 \sigma_a^2}{n_i \sigma_e^2} \right] J_{n_i} \\
 &= \frac{1}{\sigma_e^2} I_{n_i} - \frac{k_i \sigma_a^2 + kk_i (1 - k_i \sigma_a^2)}{n_i \sigma_e^2} J_{n_i} \quad (A44)
 \end{aligned}$$

and as its i, i'^{th} off-diagonal block

$[i^{th} \text{ diagonal block of } V^{-1}][i, i'^{th} \text{ off-diagonal block of } (I_N - P_3)]$

$$\begin{aligned}
 &= \left[\frac{1}{\sigma_e^2} I_{n_i} - \frac{k_i \sigma_a^2}{n_i \sigma_e^2} J_{n_i} \right] \left[-\frac{kk_{i'}}{n_{i'}} J_{n_i \times n_{i'}} \right] \\
 &= \left[-\frac{kk_{i'}}{n_{i'} \sigma_e^2} + \frac{kk_{i'} k_i \sigma_a^2}{n_{i'} \sigma_e^2} \right] J_{n_i \times n_{i'}} \\
 &= -\frac{kk_{i'} [1 - k_i \sigma_a^2]}{n_{i'} \sigma_e^2} J_{n_i \times n_{i'}} \\
 &= -\frac{kk_{i'} k_{i'}}{n_i n_{i'}} J_{n_i \times n_{i'}} \quad , \quad (A45)
 \end{aligned}$$

making use in the final step of the identity

$$(1 - k_i \sigma_a^2) = \frac{k_i \sigma_e^2}{n_i} = q_i \quad (A46)$$

which will be used frequently in what follows.

From $V_1 = \Sigma^+ J_{n_i}$ and (A44) $V_1 R_3$ has its i^{th} diagonal block

$$J_{n_i} \left[\frac{1}{\sigma_e^2} I_{n_i} - \frac{k_i \sigma_a^2 + kk_i (1 - k_i \sigma_a^2)}{n_i \sigma_e^2} J_{n_i} \right]$$

$$= \frac{(1 - kk_i)(1 - k_i \sigma_a^2)}{\sigma_e^2} J_{n_i} = \frac{k_i(1 - kk_i)}{n_i} J_{n_i} \quad (A47)$$

and from (A45) its i, i'^{th} off-diagonal block

$$J_{n_i} \left[- \frac{kk_i k_{i'}}{n_i n_{i'}} J_{n_i} x_{n_{i'}} \right] = - \frac{kk_i k_{i'}}{n_{i'}} J_{n_i} x_{n_{i'}} \quad ; \quad (A48)$$

from (A44) — (A48) $R_3 V_1 R_3$ has its i^{th} diagonal block

$[i^{th} \text{ diagonal block of } R_3][i^{th} \text{ diagonal block of } V_1 R_3]$

$$+ \sum_{i' \neq i} [i, i'^{th} \text{ off-diag. block of } R_3] [i', i^{th} \text{ off-diag. block of } V_1 R_3]$$

$$\begin{aligned} &= \left[\frac{1}{\sigma_e^2} I_{n_i} - \frac{k_i \sigma_a^2 + kk_i(1 - k_i \sigma_a^2)}{n_i \sigma_e^2} J_{n_i} \right] \left[\frac{k_i(1 - kk_i)}{n_i} J_{n_i} \right] \\ &\quad + \sum_{i' \neq i} \left[- \frac{kk_i k_{i'}}{n_i n_{i'}} J_{n_i} x_{n_{i'}} \right] \left[- \frac{kk_{i'} k_i}{n_{i'}} J_{n_{i'}} x_{n_i} \right] \\ &= \frac{k_i(1 - kk_i) - k_i^2 \sigma_a^2(1 - kk_i) - kk_i^2(1 - kk_i)(1 - k_i \sigma_a^2)}{n_i \sigma_e^2} J_{n_i} \\ &\quad + \sum_{i' \neq i} \frac{k_i^2 k_{i'}^2}{n_i^2} J_{n_i} \\ &= \frac{k_i(1 - kk_i) \frac{k_i \sigma_e^2}{n_i} - kk_i^2 \frac{k_i \sigma_e^2}{n_i}}{n_i \sigma_e^2} J_{n_i} + \frac{k_i^2 \sum_{i' \neq i} k_{i'}^2}{n_i^2} J_{n_i} \\ &= \frac{k_i^2}{n_i^2} \left(1 - 2kk_i + k^2 \sum k_i^2 \right) J_{n_i} \quad (A49) \end{aligned}$$

and from (A44) — (A48) its i, i' th off-diagonal block

$[i^{\text{th}}$ diagonal block of $R_3][i, i'$ th off-diagonal block of $V_1 R_3]$

+ $[i, i'$ th off-diagonal block of $R_3][i'^{\text{th}}$ diagonal block of $V_1 R_3]$

+ $\sum_{i'' \neq i, i'} [i, i''^{\text{th}}$ off-diag. block of $R_3][i'', i'$ th off-diag. block of $V_1 R_3]$

$$\begin{aligned}
 &= \left[\frac{1}{\sigma_e^2} I_{n_i} - \frac{k_i \sigma_a^2 + k k_i (1 - k_i \sigma_a^2)}{n_i \sigma_e^2} J_{n_i} \right] \left[- \frac{k k_i k_{i'}}{n_{i'}} J_{n_i} x_{n_{i'}} \right] \\
 &\quad + \left[- \frac{k k_i k_{i'}}{n_i n_{i'}} J_{n_i} x_{n_{i'}} \right] \left[\frac{k_i (1 - k k_{i'})}{n_{i'}} J_{n_{i'}} \right] \\
 &\quad + \sum_{i'' \neq i, i'} \left[- \frac{k k_i k_{i''}}{n_i n_{i''}} J_{n_i} x_{n_{i''}} \right] \left[- \frac{k k_{i'} k_{i''}}{n_{i'}} J_{n_{i'}} x_{n_{i''}} \right] \\
 &= \left[- \frac{k k_i k_{i'} (1 - k_i \sigma_a^2) - k^2 k_i^2 k_{i'} \frac{k_i \sigma_e^2}{n_i}}{n_{i'} \sigma_e^2} - \frac{k k_i k_{i'}^2 (1 - k k_{i'})}{n_i n_{i'}} \right. \\
 &\quad \left. + \sum_{i'' \neq i, i'} \frac{k^2 k_i k_{i'} k_{i''}^2}{n_i n_{i'}} \right] J_{n_i} x_{n_{i'}} \\
 &= \left[- \frac{k k_i k_{i'} \frac{k_i \sigma_e^2}{n_i}}{n_{i'} \sigma_e^2} - \frac{k k_i k_{i'}^2}{n_i n_{i'}} + \frac{k^2 k_i k_{i'} \Sigma k_{i'}^2}{n_i n_{i'}} \right] J_{n_i} x_{n_{i'}} \\
 &= \frac{k_i k_{i'} (k^2 \Sigma k_{i'}^2 - k [k_i + k_{i'}])}{n_i n_{i'}} J_{n_i} x_{n_{i'}} ;
 \end{aligned}$$

(A50)

and from (A47) and (A48) $(V_1 R_3)^2$ has its i^{th} diagonal block

$$\begin{aligned}
 & \left[\frac{k_i(1 - kk_i)}{n_i} J_{n_i} \right]^2 + \sum_{i' \neq i} \left[-\frac{kk_i k_{i'}}{n_{i'}} J_{n_i, x_{n_i}} \right] \left[-\frac{kk_{i'} k_i}{n_i} J_{n_i, x_{n_i}} \right] \\
 &= \left[\frac{k_i^2(1 - kk_i)^2}{n_i} + \sum_{i' \neq i} \frac{k_i^2 k_{i'}^2 k_i^2}{n_i} \right] J_{n_i} \\
 &= \frac{k_i^2 - 2kk_i^3 + k_i^2 \sum k_i^2}{n_i} J_{n_i} \\
 &= \frac{k_i^2(1 - 2kk_i + k_i^2 \sum k_i^2)}{n_i} J_{n_i} . \tag{A51}
 \end{aligned}$$

Since $V_2 = I$,

$$V_2 R_3 = R_3 . \tag{A52}$$

Then from (A44) and (A45) $R_3 V_2 R_3 = (V_2 R_3)^2 = R_3^2$ has its i^{th} diagonal block

$$\begin{aligned}
 & \left[\frac{1}{\sigma_e^2} I_{n_i} - \frac{k_i \sigma_a^2 + kk_i(1 - k_i \sigma_a^2)}{n_i \sigma_e^2} J_{n_i} \right]^2 + \sum_{i' \neq i} \left[-\frac{kk_i k_{i'}}{n_i n_{i'}} J_{n_i, x_{n_i}} \right] \left[-\frac{kk_{i'} k_i}{n_i n_{i'}} J_{n_i, x_{n_i}} \right] \\
 &= \frac{1}{\sigma_e^4} I_{n_i} + \left[\frac{-2k_i \sigma_a^2 - 2kk_i(1 - k_i \sigma_a^2) + k_i^2 \sigma_a^4 + 2kk_i^2(1 - k_i \sigma_a^2) \sigma_a^2 + k_i^2 k_i^2(1 - k_i \sigma_a^2)^2}{n_i \sigma_e^4} \right. \\
 &\quad \left. + \sum_{i' \neq i} \frac{k_i^2 k_{i'}^2 k_i^2}{n_i^2 n_{i'}} \right] J_{n_i} \\
 &= \frac{1}{\sigma_e^4} I_{n_i} + \left[\frac{-2k_i \sigma_a^2 + k_i^2 \sigma_a^4 - 2kk_i(1 - k_i \sigma_a^2) + 2kk_i^2(1 - k_i \sigma_a^2) \sigma_a^2}{n_i \sigma_e^4} \right] J_{n_i}
 \end{aligned}$$

$$\begin{aligned}
 & + \sum_{\text{all } i'} \left[\frac{k_i^2 k_{i'}^2}{n_i^2 n_{i'}} \right] J_{n_i} \\
 & = \frac{1}{\sigma_e^4} I_{n_i} + \left[\frac{(1 - k_i \sigma_a^2)^2 - 1 - 2kk_i(1 - k_i \sigma_a^2)^2}{n_i \sigma_e^4} + \frac{k_i^2 k_{i'}^2}{n_i^2} \sum \frac{k_{i'}^2}{n_{i'}} \right] J_{n_i} \\
 & = \frac{1}{\sigma_e^4} I_{n_i} + \left[-\frac{1}{n_i \sigma_e^4} + \frac{k_i^2(1 - 2kk_i)}{n_i^3} + \frac{k_i^2 k_{i'}^2}{n_i^2} \sum \frac{k_{i'}^2}{n_{i'}} \right] J_{n_i} \quad (A53)
 \end{aligned}$$

and also from (A44) and (A45) its i, i' th off-diagonal block

$$\begin{aligned}
 & \left[\frac{1}{\sigma_e^2} I_{n_i} - \frac{k_i \sigma_a^2 + kk_i(1 - k_i \sigma_a^2)}{n_i \sigma_e^2} J_{n_i} \right] \left[-\frac{kk_i k_{i'}}{n_i n_{i'}} J_{n_i, x_{n_{i'}}} \right] \\
 & + \left[-\frac{kk_i k_{i'}}{n_i n_{i'}} J_{n_i, x_{n_{i'}}} \right] \left[\frac{1}{\sigma_e^2} I_{n_{i'}} - \frac{k_{i'} \sigma_a^2 + kk_{i'}(1 - k_{i'} \sigma_a^2)}{n_{i'} \sigma_e^2} J_{n_{i'}} \right] \\
 & + \sum_{i'' \neq i \text{ or } i'} \left[-\frac{kk_i k_{i''}}{n_i n_{i''}} J_{n_i, x_{n_{i''}}} \right] \left[-\frac{kk_{i'} k_{i''}}{n_{i'} n_{i''}} J_{n_{i'}, x_{n_{i''}}} \right] \\
 & = \left[\frac{-kk_i k_{i'} + kk_i^2 k_{i'} \sigma_a^2 + k^2 k_i^2 k_{i'} (1 - k_i \sigma_a^2) - kk_{i'} k_i + kk_{i'}^2 k_i \sigma_a^2 + k^2 k_{i'}^2 k_i (1 - k_{i'} \sigma_a^2)}{n_i n_{i'} \sigma_e^2} \right. \\
 & \quad \left. + \sum_{i'' \neq i \text{ or } i'} \left[\frac{k^2 k_i k_{i'} k_{i''}^2}{n_i n_{i'} n_{i''}} \right] J_{n_i, x_{n_{i'}}} \right]
 \end{aligned}$$

$$\begin{aligned}
 &= \left[\frac{-kk_i k_i (1 - k_i \sigma_a^2) - kk_i k_i (1 - k_i \sigma_a^2)}{n_i n_i \sigma_e^2} + \sum_{\text{all } i''} \frac{k^2 k_i k_i k_{i''}^2}{n_i n_i n_{i''}} \right] J_{n_i} x_{n_i} \\
 &= \left[- \frac{kk_i^2 k_i}{n_i^2 n_i} - \frac{kk_i k_i^2}{n_i n_i^2} + \frac{k^2 k_i k_i}{n_i n_i} \sum \frac{k_i^2}{n_i} \right] J_{n_i} x_{n_i} \quad . \quad (A54)
 \end{aligned}$$

The elements of S_3 and U_3 can now be obtained. From (A51)

$$\begin{aligned}
 s_{3:11} &= \text{tr}(V_1 R_3)^2 = \sum n_i \left[\frac{k_i^2 (1 - 2kk_i + k^2 \sum k_i^2)}{n_i} \right] \\
 &= \sum k_i^2 - 2k \sum k_i^3 + k^2 \left(\sum k_i^2 \right)^2 ; \quad (A55)
 \end{aligned}$$

from (A49)

$$\begin{aligned}
 s_{3:12} &= \text{tr}(V_1 R_3 V_2 R_3) = \text{tr}(R_3 V_1 R_3) \\
 &= \sum n_i \left[\frac{k_i^2}{n_i^2} \left(1 - 2kk_i + k^2 \sum k_i^2 \right) \right] \\
 &= \sum \frac{k_i^2}{n_i} - 2k \sum \frac{k_i^3}{n_i} + k^2 \sum k_i^2 \sum \frac{k_i^2}{n_i} ; \quad (A56)
 \end{aligned}$$

and from (A53)

$$\begin{aligned}
 s_{3:22} &= \text{tr}(V_2 R_3)^3 \\
 &= \frac{N}{\sigma_e^4} - \frac{a}{\sigma_e^4} + \sum \frac{k_i^2}{n_i^2} (1 - 2kk_i) + \sum \left(\frac{k^2 k_i^2}{n_i} \sum \frac{k_i^2}{n_i} \right)
 \end{aligned}$$

$$= \frac{N - a}{\sigma_e^4} + \sum_i \frac{k_i^2}{n_i^2} - 2k \sum_i \frac{k_i^3}{n_i^2} + k^2 \left(\sum_i \frac{k_i^2}{n_i} \right)^2 \quad (A57)$$

Using (A49) and (A50) we get

$$U_{3:1} = y' R_3 V_1 R_3 y$$

$$\begin{aligned} &= \sum_i \left[\frac{k_i^2}{n_i^2} y_{i.}^2 + \sum_{\text{all } i'} \left\{ \frac{k_i k_{i'}}{n_i n_{i'}} \left[k^2 \sum_i k_i^2 - k(k_i + k_{i'}) \right] y_{i.} y_{i'.} \right\} \right] \\ &= \sum_i \left[k_i^2 \bar{y}_{i.}^2 + \sum_{\text{all } i'} \left\{ k_i k_{i'} \left[k^2 \sum_i k_i^2 - k(k_i + k_{i'}) \right] \bar{y}_{i.} \bar{y}_{i'.} \right\} \right] \\ &= \sum_i \left[k_i^2 \bar{y}_{i.}^2 + k_i \left\{ k^2 \sum_i k_i^2 \sum_i k_i \bar{y}_{i.} - k k_i \sum_i k_i \bar{y}_{i.} - k \sum_i k_i^2 \bar{y}_{i.} \right\} \bar{y}_{i.} \right] \\ &= \sum_i \left[k_i^2 \bar{y}_{i.}^2 + k^2 \sum_i k_i^2 \left(\sum_i k_i \bar{y}_{i.} \right)^2 - 2k \sum_i k_i \bar{y}_{i.} \sum_i k_i^2 \bar{y}_{i.} \right] \\ &= \sum_i k_i^2 \left[\bar{y}_{i.} - k \sum_i k_i \bar{y}_{i.} \right]^2 \quad (A58) \end{aligned}$$

while (A53) and (A54) yield

$$U_{3:2} = y' R_3 V_2 R_3 y$$

$$= \frac{1}{\sigma_e^4} \sum_i \sum_j y_{ij}^2 + \sum_i \left[\left(\frac{-1}{n_i \sigma_e^4} + \frac{k_i^2}{n_i^3} \right) y_{i.}^2 + \sum_{\text{all } i'} \left\{ \left(-\frac{k k_i^2 k_{i'}}{n_i^2 n_{i'}} - \frac{k k_i k_{i'}^2}{n_i n_{i'}^2} \right) y_{i.} y_{i'.} \right\} \right]$$

$$\begin{aligned}
& + \frac{k^2 k_i k_{i'}}{n_i n_{i'}} \left[\sum \frac{k_i^2}{n_i} y_{i.} y_{i'.} \right] \\
& = \frac{1}{\sigma_e^4} \sum \sum y_{ij}^2 + \sum_i \left[\left(\frac{-n_i}{\sigma_e^4} + \frac{k_i^2}{n_i} \right) \bar{y}_{i.}^2 + \sum_{\text{all } i'} \left\{ \left(-\frac{k k_i^2 k_{i'}}{n_i} - \frac{k k_i k_{i'}^2}{n_{i'}} \right. \right. \right. \\
& \quad \left. \left. \left. + k^2 k_i k_{i'} \right) \sum \frac{k_i^2}{n_i} \bar{y}_{i.} \bar{y}_{i'.} \right\} \right] \\
& = \frac{\text{SSE}}{\sigma_e^4} + \sum_i \left[\frac{k_i^2}{n_i} \bar{y}_{i.}^2 - \frac{k k_i^2 (\sum k_{i'} \bar{y}_{i'.}) \bar{y}_{i.}}{n_i} - k k_i \left(\sum \frac{k_{i'}^2 \bar{y}_{i'.}}{n_{i'}} \right) \bar{y}_{i.} \right. \\
& \quad \left. + k^2 k_i \left(\sum \frac{k_{i'}^2}{n_{i'}} \right) \left(\sum k_{i'} \bar{y}_{i'.} \right) \bar{y}_{i.} \right] \\
& = \frac{\text{SSE}}{\sigma_e^4} + \sum_i \frac{k_i^2}{n_i} \bar{y}_{i.}^2 - 2k \left(\sum k_{i'} \bar{y}_{i'.} \right) \left(\sum \frac{k_{i'}^2 \bar{y}_{i'.}}{n_{i'}} \right) + k^2 \left(\sum \frac{k_{i'}^2}{n_{i'}} \right) \left(\sum k_{i'} \bar{y}_{i'.} \right)^2 \\
& = \frac{\text{SSE}}{\sigma_e^4} + \sum_i \frac{k_i^2}{n_i} \left[\bar{y}_{i.} - k \sum k_{i'} \bar{y}_{i'.} \right]^2 \tag{A59}
\end{aligned}$$

where $\text{SSE} = \sum_{ij} y_{ij}^2 - \sum_i n_i \bar{y}_{i.}^2$ = the customary residual or error sum of squares for the 1-way AOV.

The estimators are then

$$\hat{\sigma}_{3e}^2 = \frac{1}{\Delta_3} [-s_{3:12} u_{3:1} + s_{3:11} u_{3:2}] \tag{A60}$$

and

$$\hat{\sigma}_{3a}^2 = \frac{1}{\Delta_3} [s_{3:22}u_{3:1} - s_{3:12}u_{3:2}] \quad (A61)$$

using (A55) through (A59) and defining $\Delta_3 = s_{3:11}s_{3:22} - s_{3:12}^2$ = the determinant of S_3 .

Using the results that $v(y'Ay) = 2\text{tr}(VA)^2$ and $\text{cov}(y'Ay, y'By) = 2\text{tr}(VAVB)$ when $y \sim N(\mu, V)$, $AX = A1 = 0$, and $BX = B1 = 0$, the variances and covariance of estimators are

$$v(\hat{\sigma}_{3e}^2) = \frac{1}{\Delta_3^2} [s_{3:12}^2 v(u_{3:1}) + s_{3:11}^2 v(u_{3:2}) - 2s_{3:11}s_{3:12}\text{cov}(u_{3:1}, u_{3:2})] \quad (A62)$$

$$v(\hat{\sigma}_{3a}^2) = \frac{1}{\Delta_3^2} [s_{3:22}^2 v(u_{3:1}) + s_{3:12}^2 v(u_{3:2}) - 2s_{3:12}s_{3:22}\text{cov}(u_{3:1}, u_{3:2})], \quad (A63)$$

$$\text{and } \text{cov}(\hat{\sigma}_{3e}^2, \hat{\sigma}_{3a}^2) = \frac{1}{\Delta_3^2} [-s_{3:12}s_{3:22}v(u_{3:1}) - s_{3:11}s_{3:12}v(u_{3:2}) + (s_{3:11}s_{3:22} + s_{3:12}^2)\text{cov}(u_{3:1}, u_{3:2})] \quad (A64)$$

However, these expressions simplify considerably. Noting that $(I - P_3)$ is idempotent and $R_3 = V^{-1}[I - P_3]$ and using (A55), (A56), and (A57), we find that under normality

$$\begin{aligned} v(u_{3:1}) &= 2\text{tr}(VR_3V_1R_3)^2 \\ &= 2\text{tr}(VV^{-1}[I - P_3]V_1R_3)^2 \\ &= 2\text{tr}([I - P_3]V_1R_3[I - P_3]V_1R_3) \\ &= 2\text{tr}(V_1R_3[I - P_3]V_1R_3(I - P_3)) \\ &= 2\text{tr}(V_1V^{-1}[I - P_3][I - P_3]V_1V^{-1}[I - P_3][I - P_3]) \end{aligned}$$

$$\begin{aligned}
 &= 2\text{tr}(V_1 V^{-1} [I - P_3] V_1 V^{-1} [I - P_3]) \\
 &= 2\text{tr}(V_1 R_3)^2 = 2s_{3:11}
 \end{aligned} \tag{A65}$$

and by analogous steps that

$$\begin{aligned}
 v(u_{3:2}) &= 2\text{tr}(VR_3 V_2 R_3)^2 \\
 &= 2\text{tr}(V_2 R_3)^2 \\
 &= 2s_{3:22}
 \end{aligned} \tag{A66}$$

$$\begin{aligned}
 \text{and } \text{cov}(u_{3:1}, u_{3:2}) &= 2\text{tr}(VR_3 V_1 R_3 VR_3 V_2 R_3) \\
 &= 2\text{tr}(V_1 R_3 V_2 R_3) = 2\text{tr}(R_3 V_1 R_3) \\
 &= 2s_{3:12}
 \end{aligned} \tag{A67}$$

Substituting (A65), (A66), and (A67) into (A62), (A63), and (A64) and simplifying gives

$$v(\hat{\sigma}_{3e}^2) = \frac{2s_{3:11}}{\Delta_3}, \tag{A68}$$

$$v(\hat{\sigma}_{3a}^2) = \frac{2s_{3:22}}{\Delta_3}, \tag{A69}$$

$$\text{and } \text{cov}(\hat{\sigma}_{3e}^2, \hat{\sigma}_{3a}^2) = \frac{-2s_{3:12}}{\Delta_3}. \tag{A70}$$

4. MINQUE for the model with $\mu \neq 0$

The elements of S_4 and U_4 and the estimators of the variance components may be obtained from the preceding MINQUE results by setting

$\sigma_e^2 = \sigma_a^2 = 1$. A convenient notation is

$$\ell_i = \frac{n_i}{1 + n_i} \text{ and } \ell = \frac{1}{\sum \ell_i} \quad (\text{A71})$$

from which

$$\ell \sum \ell_i = 1 \quad ,$$

$$1 - \ell_i = \frac{1}{1 + n_i} \quad ,$$

$$\ell_i = n_i(1 - \ell_i) \quad ,$$

and

$$n_i \ell_i = n_i - \ell_i$$

are useful identities.

From (A55) through (A59) we get

$$s_{4:11} = \sum \ell_i^2 - 2\ell \sum \ell_i^3 + \ell^2 \left(\sum \ell_i^2 \right)^2 \quad , \quad (\text{A72})$$

$$\begin{aligned} s_{4:12} &= \sum \frac{\ell_i^2}{n_i} - 2\ell \sum \frac{\ell_i^3}{n_i} + \ell^2 \sum \ell_i^2 \sum \frac{\ell_i^2}{n_i} \\ &= \sum \ell_i(1 - \ell_i) - 2\ell \sum \ell_i^2(1 - \ell_i) + \ell^2 \sum \ell_i^2 \sum \ell_i(1 - \ell_i) \\ &= \sum \ell_i - \sum \ell_i^2 - \ell \sum \ell_i^2 + 2\ell \sum \ell_i^3 - \ell^2 \left(\sum \ell_i^2 \right)^2 \\ &= -s_{4:11} - \ell \sum \ell_i^2 + \sum \ell_i \quad , \end{aligned} \quad (\text{A73})$$

$$\begin{aligned}
 s_{4:22} &= N - a + \sum \frac{l_i^2}{n_i^2} - 2\ell \sum \frac{l_i^3}{n_i^2} + \ell^2 \left(\sum \frac{l_i^2}{n_i} \right)^2 \\
 &= N - a + \sum (1 - l_i)^2 - 2\ell \sum l_i(1 - l_i)^2 + \ell^2 \left(\sum l_i[1 - l_i] \right)^2 \\
 &= N - a + a - 2 \sum l_i + \sum l_i^2 - 2 + 4\ell \sum l_i^2 - 2\ell \sum l_i^3 + 1 \\
 &\quad - 2\ell \sum l_i^2 + \ell^2 \left(\sum l_i^2 \right)^2 \\
 &= s_{4:11} + N - 1 - 2 \sum l_i + 2\ell \sum l_i^2, \tag{A74}
 \end{aligned}$$

$$u_{4:1} = \sum l_i^2 \left[\bar{y}_{i\cdot} - \ell \sum l_i \bar{y}_{i\cdot} \right]^2, \tag{A75}$$

$$\begin{aligned}
 \text{and } u_{4:2} &= \text{SSE} + \sum \frac{l_i^2}{n_i} \left[\bar{y}_{i\cdot} - \ell \sum l_i \bar{y}_{i\cdot} \right]^2 \\
 &= \text{SSE} + \sum l_i(1 - l_i) \left[\bar{y}_{i\cdot} - \ell \sum l_i \bar{y}_{i\cdot} \right]^2 \\
 &= -u_{4:1} + \text{SSE} + \sum l_i \left[\bar{y}_{i\cdot} - \ell \sum l_i \bar{y}_{i\cdot} \right]^2. \tag{A76}
 \end{aligned}$$

The estimators are

$$\hat{\sigma}_{4e}^2 = \frac{1}{\Delta_4} [-s_{4:12}u_{4:1} + s_{4:11}u_{4:2}] \tag{A77}$$

$$\text{and } \hat{\sigma}_{4a}^2 = \frac{1}{\Delta_4} [s_{4:22}u_{4:1} - s_{4:12}u_{4:2}] \tag{A78}$$

using (A72) through (A76) and defining $\Delta_4 = s_{4:11}s_{4:22} - s_{4:12}^2$ the deter-

minant of S^4 . Δ_4 can also be found as $\Delta_4 = (N - 1)s_{4:11} - [\ell \sum \ell_i^2 - \ell^{-1}]^2$ but this form is not very useful since one would still calculate $s_{4:11}$, $s_{4:12}$, and $s_{4:22}$ for (A77) and (A78).

The variances and covariance of the estimators are given by

$$v(\hat{\sigma}_{4e}^2) = \frac{1}{\Delta_4^2} [s_{4:12}^2 v(u_{4:1}) + s_{4:11}^2 v(u_{4:2}) - 2s_{4:11}s_{4:12} \text{cov}(u_{4:1}, u_{4:2})] , \quad (\text{A79})$$

$$v(\hat{\sigma}_{4a}^2) = \frac{1}{\Delta_4^2} [s_{4:22}^2 v(u_{4:1}) + s_{4:12}^2 v(u_{4:2}) - 2s_{4:12}s_{4:22} \text{cov}(u_{4:1}, u_{4:2})] , \quad (\text{A80})$$

$$\text{and } \text{cov}(\hat{\sigma}_{4e}^2, \hat{\sigma}_{4a}^2) = \frac{1}{\Delta_4^2} [-s_{4:12}s_{4:22} v(u_{4:1}) - s_{4:11}s_{4:12} v(u_{4:2}) + (s_{4:11}s_{4:22} + s_{4:12}^2) \text{cov}(u_{4:1}, u_{4:2})] . \quad (\text{A81})$$

Expressions for $v(u_{4:1})$, $v(u_{4:2})$, and $\text{cov}(u_{4:1}, u_{4:2})$ are not obtainable from the MIVQUE case, so are derived below.

We can get $R_4 V_1 R_4$ and $R_4 V_2 R_4$ from the analogous MIVQUE expressions by setting $\sigma_e^2 = \sigma_a^2 = 1$. From (A49) and (A50) $R_4 V_1 R_4$ has its i^{th} diagonal block

$$\begin{aligned} & \frac{\ell_i^2}{n_i^2} \left(1 - 2\ell\ell_i + \ell^2 \sum \ell_i^2 \right) J_{n_i} \\ & = (1 - \ell_i)^2 \left(1 - 2\ell\ell_i + \ell^2 \sum \ell_i^2 \right) J_{n_i} \end{aligned} \quad (\text{A82})$$

and its i, i'^{th} off-diagonal block

$$\frac{\ell_i \ell_{i'} (\ell^2 \sum \ell_i^2 - \ell [\ell_i + \ell_{i'}])}{n_i n_{i'}} J_{n_i \times n_{i'}} \\ = (1 - \ell_i)(1 - \ell_{i'}) (\ell^2 \sum \ell_i^2 - \ell [\ell_i + \ell_{i'}]) J_{n_i \times n_{i'}} \quad (A83)$$

From (A53) and (A54) $R_4 V_2 R_4$ has its i^{th} diagonal block

$$I_{n_i} + \left[\frac{-1}{n_i} + \frac{\ell_i^2 (1 - 2\ell \ell_i)}{n_i^3} + \frac{\ell^2 \ell_i^2}{n_i^2} \sum \frac{\ell_i^2}{n_i} \right] J_{n_i} \\ = I_{n_i} + \left[\frac{-1 + (1 - \ell_i)^2 (1 - 2\ell \ell_i) + \ell^2 \ell_i (1 - \ell_i) \sum \ell_i (1 - \ell_i)}{n_i} \right] J_{n_i} \\ = I_{n_i} + \left[\frac{-1 + (1 - \ell_i)(1 - 2\ell \ell_i) + 2\ell \ell_i (1 - \ell_i) + \ell_i (1 - \ell_i) (-1 - \ell + 2\ell \ell_i - \ell^2 \sum \ell_i^2)}{n_i} \right] J_{n_i} \\ = I_{n_i} + \left[\frac{-\ell_i + \ell_i (1 - \ell_i) (-1 - \ell + 2\ell \ell_i - \ell^2 \sum \ell_i^2)}{n_i} \right] J_{n_i} \\ = I_{n_i} + \left[-(1 - \ell_i) + (1 - \ell_i)^2 (-1 - \ell + 2\ell \ell_i - \ell^2 \sum \ell_i^2) \right] J_{n_i} \quad (A84)$$

and its i, i'^{th} off-diagonal block

$$\left[-\frac{\ell \ell_i^2 \ell_{i'}}{n_i^2 n_{i'}} - \frac{\ell \ell_i \ell_{i'}^2}{n_i n_{i'}^2} + \frac{\ell^2 \ell_i \ell_{i'}}{n_i n_{i'}} \sum \frac{\ell_i^2}{n_i} \right] J_{n_i \times n_{i'}} \\ = (1 - \ell_i)(1 - \ell_{i'}) \left[-\ell (1 - \ell_i) - \ell (1 - \ell_{i'}) + \ell^2 \sum \ell_i (1 - \ell_i) \right] J_{n_i \times n_{i'}}$$

$$= (1 - \ell_i)(1 - \ell_{i'}) \left[-\ell + \ell(\ell_i + \ell_{i'}) - \ell^2 \sum \ell_i^2 \right] J_{n_i x_{n_i}}, \quad (A85)$$

Now $v(u_{4:1}) = 2\text{tr}(VR_4 V_1 R_4)^2$ where from (A82) and (A83) and k_i defined by (A43) $VR_4 V_1 R_4$ has its i^{th} diagonal block

$$\begin{aligned} & \left[\sigma_e^2 I_{n_i} + \sigma_a^2 J_{n_i} \right] \left[(1 - \ell_i)^2 (1 + \ell^2 \sum \ell_i^2 - 2\ell\ell_i) J_{n_i} \right] \\ &= (\sigma_e^2 + n_i \sigma_a^2) (1 - \ell_i)^2 (1 + \ell^2 \sum \ell_i^2 - 2\ell\ell_i) J_{n_i} \\ &= \frac{n_i}{k_i} (1 - \ell_i)^2 (1 + \ell^2 \sum \ell_i^2 - 2\ell\ell_i) J_{n_i} \\ &= \frac{\ell_i (1 - \ell_i)}{k_i} (1 + \ell^2 \sum \ell_i^2 - 2\ell\ell_i) J_{n_i} \end{aligned} \quad (A86)$$

and its i, i'^{th} off-diagonal block

$$\begin{aligned} & \left[\sigma_e^2 I_{n_i} + \sigma_a^2 J_{n_i} \right] \left[(1 - \ell_i)(1 - \ell_{i'}) (\ell^2 \sum \ell_i^2 - \ell[\ell_i + \ell_{i'}]) J_{n_i x_{n_i}} \right] \\ &= \frac{n_i}{k_i} (1 - \ell_i)(1 - \ell_{i'}) (\ell^2 \sum \ell_i^2 - \ell[\ell_i + \ell_{i'}]) J_{n_i x_{n_i}}, \\ &= \frac{\ell_i (1 - \ell_{i'})}{k_i} (\ell^2 \sum \ell_i^2 - \ell[\ell_i + \ell_{i'}]) J_{n_i x_{n_i}}, \end{aligned} \quad (A87)$$

From (A86) and (A87) $(VR_4 V_1 R_4)^2$ has its i^{th} diagonal block

$$\begin{aligned} & \left[\frac{\ell_i^3 (1 - \ell_i)}{k_i^2} (1 + \ell^2 \sum \ell_i^2 - 2\ell\ell_i)^2 + \sum_{i' \neq i} \frac{\ell_i \ell_{i'}^2 (1 - \ell_i)}{k_i k_{i'}} (\ell^2 \sum \ell_i^2 - \ell[\ell_i + \ell_{i'}])^2 \right] J_{n_i} \\ &= \left[\frac{\ell_i^3 (1 - \ell_i)}{k_i^2} (1 + 2\ell^2 \sum \ell_i^2 - 4\ell\ell_i) \right. \end{aligned}$$

$$\begin{aligned}
& + \frac{\ell_i(1 - \ell_i)}{k_i} \sum_{\text{all } i'} \frac{\ell_{i'}^2}{k_{i'}} \left\{ \ell^4 \left(\sum \ell_{i'}^2 \right)^2 + \ell^2 \ell_{i'}^2 + \ell^2 \ell_{i'}^2 \right. \\
& \quad \left. - 2\ell^3 \ell_{i'} \sum \ell_{i'}^2 - 2\ell^3 \ell_{i'} \sum \ell_{i'}^2 + 2\ell^2 \ell_{i'} \ell_{i'} \right\} J_{n_i} \\
& = \left[\frac{\ell_i^3(1 - \ell_i)}{k_i^2} \left(1 + 2\ell^2 \sum \ell_{i'}^2 - 4\ell \ell_{i'} \right) \right. \\
& \quad + \frac{\ell_i(1 - \ell_i)}{k_i} \left\{ \ell^4 \left(\sum \ell_{i'}^2 \right)^2 \sum \frac{\ell_{i'}^2}{k_{i'}} + \ell^2 \ell_{i'}^2 \sum \frac{\ell_{i'}^2}{k_{i'}} + \ell^2 \sum \frac{\ell_{i'}^4}{k_{i'}} \right. \\
& \quad \left. \left. - 2\ell^3 \ell_{i'} \sum \ell_{i'}^2 \sum \frac{\ell_{i'}^2}{k_{i'}} - 2\ell^3 \sum \ell_{i'}^2 \sum \frac{\ell_{i'}^3}{k_{i'}} + 2\ell^2 \ell_{i'} \sum \frac{\ell_{i'}^3}{k_{i'}} \right\} \right] J_{n_i} \quad . \quad (A88)
\end{aligned}$$

Then $v(u_{4:1}) = 2\text{tr}(VR_4V_1R_4)^2$

$$\begin{aligned}
& = 2 \left[\sum \frac{\ell_i^4}{k_i^2} + 2\ell^2 \sum \ell_{i'}^2 \sum \frac{\ell_{i'}^4}{k_{i'}^2} - 4\ell \sum \frac{\ell_i^5}{k_i^2} + \ell^4 \left(\sum \ell_{i'}^2 \right)^2 \left(\sum \frac{\ell_{i'}^2}{k_{i'}} \right)^2 \right. \\
& \quad \left. + 2\ell^2 \sum \frac{\ell_i^2}{k_i} \sum \frac{\ell_{i'}^4}{k_{i'}} - 4\ell^3 \sum \ell_{i'}^2 \sum \frac{\ell_{i'}^2}{k_{i'}} \sum \frac{\ell_{i'}^3}{k_{i'}} + 2\ell^2 \left(\sum \frac{\ell_{i'}^3}{k_{i'}} \right)^2 \right] \quad . \quad (A89)
\end{aligned}$$

Similarly, $v(u_{4:2}) = 2\text{tr}(VR_4V_2R_4)^2$ where from (A84) and (A85) $VR_4V_2R_4$ has its i^{th} diagonal block

$$\begin{aligned}
& \left[\sigma_{e n_i}^2 I_{n_i} + \sigma_{a n_i}^2 J_{n_i} \right] \left[I_{n_i} + \left\{ -(1 - \ell_i) + (1 - \ell_i)^2(-1 - \ell + 2\ell \ell_{i'} - \ell^2 \sum \ell_{i'}^2) \right\} J_{n_i} \right] \\
& = \sigma_{e n_i}^2 I_{n_i} + \sigma_{a n_i}^2 J_{n_i} + \frac{n_i}{k_i} \left\{ -(1 - \ell_i) + (1 - \ell_i)^2(-1 - \ell + 2\ell \ell_{i'} - \ell^2 \sum \ell_{i'}^2) \right\} J_{n_i} \\
& = \sigma_{e n_i}^2 I_{n_i} + \sigma_{a n_i}^2 J_{n_i} + \frac{1}{k_i} \left\{ -\ell_i + \ell_i(1 - \ell_i)(-1 - \ell + 2\ell \ell_{i'} - \ell^2 \sum \ell_{i'}^2) \right\} J_{n_i} \quad (A90)
\end{aligned}$$

and its i, i' th off-diagonal block

$$\begin{aligned}
 & \left[\sigma_e^2 I_{n_i} + \sigma_a^2 J_{n_i} \right] \left[(1 - l_i)(1 - l_{i'})(-l + l[l_i + l_{i'}] - l^2 \sum l_i^2) J_{n_i, x_{n_i}} \right] \\
 &= \frac{n_i}{k_i} (1 - l_i)(1 - l_{i'})(-l + l[l_i + l_{i'}] - l^2 \sum l_i^2) J_{n_i, x_{n_i}}, \\
 &= \frac{l_i}{k_i} (1 - l_{i'})(-l + l[l_i + l_{i'}] - l^2 \sum l_i^2) J_{n_i, x_{n_i}}, \quad (A91)
 \end{aligned}$$

So from (A90) and (A91) $(VR_4 V_2 R_4)^2$ has its i th diagonal block

$$\begin{aligned}
 & \left[\sigma_e^2 I_{n_i} + \sigma_a^2 J_{n_i} + \left\{ \frac{-l_i - l_i(1 - l_i)}{k_i} + \frac{l_i(1 - l_i)}{k_i} (-l + 2ll_i - l^2 \sum l_i^2) \right\} J_{n_i} \right]^2 \\
 &+ \sum_{i' \neq i} \frac{l_i l_{i'}^2}{k_i k_{i'}} (1 - l_i)(-l + l[l_i + l_{i'}] - l^2 \sum l_i^2)^2 J_{n_i} \\
 &= \left(\sigma_e^2 I_{n_i} + \sigma_a^2 J_{n_i} \right)^2 + \left[\frac{n_i(-l_i - l_i[1 - l_i])^2}{k_i^2} \right. \\
 &+ 2 \frac{n_i}{k_i} \left\{ \frac{-l_i + l_i(1 - l_i)(-1 - l + 2ll_i - l^2 \sum l_i^2)}{k_i} \right\} \\
 &+ 2 \frac{l_i^2(-l_i - l_i[1 - l_i])}{k_i^2} (-l + 2ll_i - l^2 \sum l_i^2) \\
 &+ \sum_{\text{all } i'} \frac{l_i l_{i'}^2}{k_i k_{i'}} (1 - l_i)(-l + l[l_i + l_{i'}] - l^2 \sum l_i^2)^2 \left. \right] J_{n_i} \\
 &= \left(\sigma_e^2 I_{n_i} + \sigma_a^2 J_{n_i} \right)^2 \\
 &+ \left[\frac{n_i l_i^2 + 2l_i^3 + l_i^3(1 - l_i) - 2n_i l_i + 2l_i^2(-1 - l + 2ll_i - l^2 \sum l_i^2) + 2l_i^3(-1 - [1 - l_i])(-l + 2ll_i - l^2 \sum l_i^2)}{k_i^2} \right] J_{n_i}
 \end{aligned}$$

$$\begin{aligned}
& + \sum_{\text{all } i'} \frac{l_i l_{i'}^2}{k_i k_{i'}} (1 - l_i) (-l + l[l_i + l_{i'}] - l^2 \sum l_i^2)^2 J_{n_i} \\
& = \left(\sigma_{e n_i}^2 + \sigma_{a n_i}^2 \right)^2 \\
& + \left[\frac{n_i l_i^2 + 3l_i^3 - l_i^4 - 2n_i l_i - 2l_i^2 + 2l_i^2(1 - l_i)^2(-l + 2ll_i - l^2 \sum l_i^2)}{k_i^2} \right. \\
& + \sum_{\text{all } i'} \frac{l_i l_{i'}^2}{k_i k_{i'}} (1 - l_i) (-l + l[l_i + l_{i'}] - l^2 \sum l_i^2)^2 J_{n_i} \\
& = \sigma_{e n_i}^4 + 2\sigma_{e a n_i}^2 \sigma_{a n_i}^2 + n_i \sigma_{a n_i}^4 \\
& + \left[\frac{-n_i + l_i(1 - l_i)^2(1 - l_i - 2ll_i + 4ll_i^2 - 2l^2 l_i \sum l_i^2)}{k_i^2} \right. \\
& + \frac{l_i(1 - l_i)}{k_i} \left\{ l^2 \sum \frac{l_i^2}{k_i} + l^2 l_i^2 \sum \frac{l_i^2}{k_i} + l^2 \sum \frac{l_i^4}{k_i} + l^4 \left(\sum l_i^2 \right)^2 \sum \frac{l_i^2}{k_i} \right. \\
& - 2l^2 l_i \sum \frac{l_i^2}{k_i} - 2l^2 \sum \frac{l_i^3}{k_i} + 2l^3 \sum l_i^2 \sum \frac{l_i^2}{k_i} + 2l^2 l_i \sum \frac{l_i^3}{k_i} \\
& \left. \left. - 2l^3 l_i \sum l_i^2 \sum \frac{l_i^2}{k_i} - 2l^3 \sum l_i^2 \sum \frac{l_i^3}{k_i} \right\} \right] J_{n_i} \tag{A92}
\end{aligned}$$

using the identity

$$n_i l_i^2 + 3l_i^3 - l_i^4 - 2n_i l_i - 2l_i^2 = -n_i + l_i(1 - l_i)^3$$

to obtain the last equality.

From (A92) and then (A89) and (A95) we get

$$v(u_{4:2}) = 2\text{tr}(VR_4 V_a R_4)^2$$

$$\begin{aligned}
&= 2 \left[N\sigma_e^4 + 2N\sigma_e^2\sigma_a^2 + \sum n_i^2\sigma_a^4 - \sum \frac{n_i^2}{k_i^2} \right. \\
&\quad + \sum \frac{\ell_i^2}{k_i^2} - 2 \sum \frac{\ell_i^3}{k_i^2} - 2\ell \sum \frac{\ell_i^3}{k_i^2} + 6\ell \sum \frac{\ell_i^4}{k_i^2} \\
&\quad - 2\ell^2 \sum \ell_i^2 \sum \frac{\ell_i^3}{k_i^2} + \sum \frac{\ell_i^4}{k_i^2} - 4\ell \sum \frac{\ell_i^5}{k_i^2} \\
&\quad + 2\ell^2 \sum \ell_i^2 \sum \frac{\ell_i^4}{k_i^2} + \ell^2 \left(\sum \frac{\ell_i^2}{k_i} \right)^2 \\
&\quad + 2\ell^2 \sum \frac{\ell_i^2}{k_i} \sum \frac{\ell_i^4}{k_i} + \ell^4 \left(\sum \ell_i^2 \right)^2 \left(\sum \frac{\ell_i^2}{k_i} \right)^2 \\
&\quad - 4\ell^2 \sum \frac{\ell_i^2}{k_i} \sum \frac{\ell_i^3}{k_i} + 2\ell^3 \sum \ell_i^2 \left(\sum \frac{\ell_i^2}{k_i} \right)^2 \\
&\quad \left. + 2\ell^2 \left(\sum \frac{\ell_i^2}{k_i} \right)^2 - 4\ell^3 \sum \ell_i^2 \sum \frac{\ell_i^2}{k_i} \sum \frac{\ell_i^3}{k_i} \right] \\
&= 2 \left[-\frac{1}{2} v(u_{4:1}) - \text{cov}(u_{4:1}, u_{4:2}) + N\sigma_e^4 \right. \\
&\quad + 2N\sigma_e^2\sigma_a^2 + \sum n_i^2\sigma_a^4 - \sum \frac{n_i^2}{k_i^2} + \sum \frac{\ell_i^2}{k_i^2} \\
&\quad \left. - 2\ell \sum \frac{\ell_i^3}{k_i^2} + \ell^2 \left(\sum \frac{\ell_i^2}{k_i} \right)^2 \right] . \tag{A93}
\end{aligned}$$

Finally, $\text{cov}(u_{4:1}, u_{4:2}) = 2\text{tr}(VR_4V_1R_4VR_4V_2R_4)$ where from (A86), (A87), (A90), and (A91) $VR_4V_1R_4VR_4V_2R_4$ has its i^{th} diagonal block

$$\left[\frac{\ell_i(1-\ell_i)}{k_i} \left(1 + \ell^2 \Sigma \ell_i^2 - 2\ell \ell_i \right) J_{n_i} \right] \left[\sigma_e^2 I_{n_i} + \sigma_a^2 J_{n_i} + \frac{1}{k_i} \left\{ -\ell_i + \ell_i(1-\ell_i)(-1-\ell + 2\ell \ell_i - \ell^2 \Sigma \ell_i^2) \right\} J_{n_i} \right]$$

$$\begin{aligned}
& + \sum_{i' \neq i} \left[\frac{\ell_i}{k_i} (1 - \ell_{i'}) (\ell^2 \Sigma \ell_i^2 - \ell [\ell_i + \ell_{i'}]) J_{n_i, x_{n_i}} \left[\frac{\ell_{i'}}{k_{i'}} (1 - \ell_{i'}) (-\ell + \ell [\ell_i + \ell_{i'}]) \right. \right. \\
& \quad \left. \left. - \ell^2 \Sigma \ell_i^2 \right) J_{n_i, x_{n_i}} \right] \\
& = \left[\frac{\ell_i^2}{k_i^2} (1 + \ell^2 \Sigma \ell_i^2 - 2 \ell \ell_i) + \frac{\ell_i^2}{k_i^2} (1 + \ell^2 \Sigma \ell_i^2 - 2 \ell \ell_i) \left\{ -\ell_i + \ell_{i'} (1 - \ell_{i'}) (-1 - \ell + 2 \ell \ell_i - \ell^2 \Sigma \ell_i^2) \right\} \right. \\
& \quad \left. + \sum_{i' \neq i} \frac{\ell_i \ell_{i'}^2}{k_i k_{i'}} (1 - \ell_{i'}) (\ell^2 \Sigma \ell_i^2 - \ell [\ell_i + \ell_{i'}]) (-\ell + \ell [\ell_i + \ell_{i'}] - \ell^2 \Sigma \ell_i^2) \right] J_{n_i} \\
& = \left[\frac{\ell_i^2}{k_i^2} \left\{ (1 + \ell^2 \Sigma \ell_i^2 - 2 \ell \ell_i) + (-\ell_i + \ell_{i'} [1 - \ell_{i'}] [-1 - \ell + 2 \ell \ell_i - \ell^2 \Sigma \ell_i^2]) \right. \right. \\
& \quad \left. \left. + (\ell^2 \Sigma \ell_i^2 - 2 \ell \ell_i) (-\ell_i - \ell_{i'} [1 - \ell_{i'}]) \right\} + \sum_{\text{all } i'} \frac{\ell_i \ell_{i'}^2}{k_i k_{i'}} (1 - \ell_{i'}) (\ell^2 \Sigma \ell_i^2 - \ell [\ell_i + \ell_{i'}]) (-\ell \right. \\
& \quad \left. + \ell [\ell_i + \ell_{i'}] - \ell^2 \Sigma \ell_i^2) \right] J_{n_i} \\
& = \left[\frac{\ell_i^2}{k_i^2} \left\{ (1 - \ell_i) (1 - \ell_i - \ell \ell_i + [\ell^2 \Sigma \ell_i^2 - 2 \ell \ell_i] [1 - 2 \ell_i]) \right\} \right. \\
& \quad + \frac{\ell_i (1 - \ell_i)}{k_i} \left\{ -\ell^3 \Sigma \ell_i^2 \Sigma \frac{\ell_i^2}{k_i} + 2 \ell^3 \ell_i \Sigma \ell_i^2 \Sigma \frac{\ell_i^2}{k_i} + 2 \ell^3 \Sigma \ell_i^2 \Sigma \frac{\ell_i^3}{k_i} - \ell^4 (\Sigma \ell_i^2)^2 \Sigma \frac{\ell_i^2}{k_i} \right. \\
& \quad \left. + \ell^2 \ell_i \Sigma \frac{\ell_i^2}{k_i} - \ell^2 \ell_i^2 \Sigma \frac{\ell_i^2}{k_i} - 2 \ell^2 \ell_i \Sigma \frac{\ell_i^3}{k_i} + \ell^2 \Sigma \frac{\ell_i^3}{k_i} - \ell^2 \Sigma \frac{\ell_i^4}{k_i} \right\} \right] J_{n_i} . \tag{A94}
\end{aligned}$$

From (A94) and then (A89)

$$\begin{aligned}
\text{cov}(u_{4:1}, u_{4:2}) &= 2 \text{tr}(VR_4 V_1 R_4 VR_4 V_2 R_4) \\
&= 2 \left[\sum \frac{\ell_i^3}{k_i^2} - \sum \frac{\ell_i^4}{k_i^2} - 3 \ell \sum \frac{\ell_i^4}{k_i^2} + \ell^2 \sum \ell_i^2 \sum \frac{\ell_i^3}{k_i^2} \right]
\end{aligned}$$

$$\begin{aligned}
 & - 2\ell^2 \sum \ell_i^2 \sum \frac{\ell_i^4}{k_i^2} + 4\ell \sum \frac{\ell_i^5}{k_i^2} - \ell^3 \sum \ell_i^2 \left(\sum \frac{\ell_i^2}{k_i} \right)^2 \\
 & + 4\ell^3 \sum \ell_i^2 \sum \frac{\ell_i^2}{k_i} \sum \frac{\ell_i^3}{k_i} - \ell^4 \left(\sum \ell_i^2 \right)^2 \left(\sum \frac{\ell_i^2}{k_i} \right)^2 \\
 & + 2\ell^2 \sum \frac{\ell_i^2}{k_i} \sum \frac{\ell_i^3}{k_i} - 2\ell^2 \sum \frac{\ell_i^2}{k_i} \sum \frac{\ell_i^4}{k_i} - 2\ell^2 \left(\sum \frac{\ell_i^3}{k_i} \right)^2 \\
 & = 2 \left[-\frac{1}{3} v(u_{4:1}) + \sum \frac{\ell_i^3}{k_i^2} - 3\ell \sum \frac{\ell_i^4}{k_i^2} + \ell^2 \sum \ell_i^2 \sum \frac{\ell_i^3}{k_i^2} \right. \\
 & \quad \left. - \ell^3 \sum \ell_i^2 \left(\sum \frac{\ell_i^2}{k_i} \right)^2 + 2\ell^2 \sum \frac{\ell_i^2}{k_i} \sum \frac{\ell_i^3}{k_i} \right] \quad (A95)
 \end{aligned}$$

II. Simplification under balance of the MINQUE and MIVQUE estimators and their variances and covariances

Under the condition of balance, i.e. $n_1 = n_2 = \dots = n_a = n$, we can show that the estimators for MINQUE and MIVQUE reduce to the usual AOV estimators. The variances and covariances simplify correspondingly.

1. MIVQUE with $\mu \equiv 0$

When $n_1 = n_2 = \dots = n_a = n$ and therefore for q_1 defined in (A7) $q_1 = q_2 = \dots = q_a = \bar{q}$, (A8) through (A12) give

$$s_{1:11} = \frac{an^2\bar{q}^2}{\sigma_e^4}, \quad (A96)$$

$$s_{1:12} = \frac{an\bar{q}^2}{\sigma_e^4}, \quad (A97)$$

$$s_{1:22} = \frac{a\bar{q}^2 + N - a}{\sigma_e^4}, \quad (A98)$$

$$u_{1:1} = \frac{\bar{q}^2 \sum y_{i.}^2}{\sigma_e^4}, \quad (A99)$$

and

$$\begin{aligned} u_{1:2} &= \frac{\sum \sum y_{ij}^2 - \sum \frac{y_{i.}^2}{n} + \bar{q}^2 \sum \frac{y_{i.}^2}{n}}{\sigma_e^4} \\ &= \frac{SSE + \frac{\bar{q}^2}{n} \sum y_{i.}^2}{\sigma_e^4} \end{aligned} \quad (A100)$$

Then from (A96), (A97), and (A98)

$$\begin{aligned} \Delta_1 &= s_{1:11}s_{1:22} - s_{1:12}^2 \\ &= \frac{(N - a)s_{1:11}}{\sigma_e^4} \end{aligned} \quad (A101)$$

Substituting (A96) through (A101) into (A13) and (A14) gives

$$\begin{aligned}\hat{\sigma}_{1e}^2 &= \frac{\sigma_e^4}{(N-a)s_{1:11}} \left[-\left(\frac{an\bar{q}^2}{\sigma_e^4}\right)\left(\frac{\bar{q}^2 \Sigma y_{i.}^2}{\sigma_e^4}\right) + \left(\frac{an^2 \bar{q}^2}{\sigma_e^4}\right)\left(\frac{SSE + \frac{\bar{q}^2}{n} \Sigma y_{i.}^2}{\sigma_e^4}\right) \right] \\ &= \frac{s_{1:11} SSE}{(N-a)s_{1:11}} = \frac{SSE}{N-a} = MSE\end{aligned}\quad (A102)$$

$$\begin{aligned}\text{and } \hat{\sigma}_{1a}^2 &= \frac{\sigma_e^4}{(N-a)s_{1:11}} \left[\left(\frac{a\bar{q}^2 + N-a}{\sigma_e^4}\right)\left(\frac{\bar{q}^2 \Sigma y_{i.}^2}{\sigma_e^4}\right) - \left(\frac{an\bar{q}^2}{\sigma_e^4}\right)\left(\frac{SSE + \frac{\bar{q}^2}{n} \Sigma y_{i.}^2}{\sigma_e^4}\right) \right] \\ &= \frac{\sigma_e^4}{(N-a)s_{1:11}} \left[\frac{(N-a)\bar{q}^2 \Sigma y_{i.}^2}{\sigma_e^8} - \frac{an\bar{q}^2 SSE}{\sigma_e^8} \right] \\ &= \frac{\sigma_e^4}{(N-a)s_{1:11}} \left[\frac{(N-a)an^2 \bar{q}^2 \Sigma \frac{y_{i.}^2}{n}}{an\sigma_e^8} - \frac{an^2 \bar{q}^2 SSE}{n\sigma_e^8} \right] \\ &= \frac{\sigma_e^4}{(N-a)s_{1:11}} \left[\frac{(N-a)s_{1:11} SSA}{an\sigma_e^4} - \frac{s_{1:11} SSE}{n\sigma_e^4} \right] \\ &= \frac{1}{n} \left[\frac{SSA}{a} - \frac{SSE}{N-a} \right] \\ &= \frac{1}{n} [MSA - MSE]\end{aligned}\quad (A103)$$

where SSE and SSA are respectively the error and treatment sums of squares for the 1-way AOV with $\mu = 0$, MSE and MSA are the corresponding mean squares, and (A102) and (A103) are the customary AOV estimators of the variance components.

Writing (A23), (A24), and (A25) in terms of (A96) through (A101) gives

$$v(\hat{\sigma}_{1e}^2) = \frac{2s_{1:11}}{\Delta_1}$$

$$= \frac{2\sigma_e^4 s_{1:11}}{(N - a)s_{1:11}} = \frac{2\sigma_e^4}{N - a}, \quad (A104)$$

$$\begin{aligned} v(\hat{\sigma}_{1a}^2) &= \frac{2s_{1:22}}{\Delta_1} \\ &= \frac{2\sigma_e^4 s_{1:22}}{(N - a)s_{1:11}} \\ &= \frac{2\sigma_e^4 [a\bar{q}^2 + N - a]}{(N - a)an^2\bar{q}^2} \\ &= \frac{2\sigma_e^4}{n^2(N - a)} + \frac{2\sigma_e^4}{an^2\bar{q}^2} \\ &= \frac{2\sigma_e^4}{n^2(N - a)} + \frac{2(\sigma_e^2 + n\sigma_a^2)^2}{nN}, \end{aligned} \quad (A105)$$

since $\bar{q} = \frac{\sigma_e^2}{\sigma_e^2 + n\sigma_a^2}$ from (A7) and $N = an$, and

$$\begin{aligned} \text{cov}(\hat{\sigma}_{1e}^2, \hat{\sigma}_{1a}^2) &= \frac{-2s_{1:12}}{\Delta_1} \\ &= \frac{-2\sigma_e^4 s_{1:12}}{(N - a)s_{1:11}} \\ &= \frac{-2\sigma_e^4 an\bar{q}^2}{(N - a)an^2\bar{q}^2} \\ &= \frac{-2\sigma_e^4}{n(N - a)}. \end{aligned} \quad (A106)$$

2. MINQUE with $\mu \equiv 0$

Since the MINQUE estimators can be regarded as a special case of MIVQUE,

the simplifications completed at (A102) and (A103) apply to MINQUE. We next show that under balance the variances and covariance of the MINQUE estimators reduce to (A104), (A105) and (A106).

Under balance $w_1 = w_2 = \dots = w_a = \bar{w}$, using w_i of (A27). Then setting $\sigma_e^2 = \sigma_a^2 = 1$ in (A96), (A97), (A98), and (A101)

$$s_{2:11} = an^2\bar{w}^2, \quad (A107)$$

$$s_{2:12} = an\bar{w}^2, \quad (A108)$$

$$s_{2:22} = a\bar{w}^2 + N - a, \quad (A109)$$

and

$$\Delta_2 = s_{2:11}s_{2:22} - s_{2:12}^2$$

$$= (N - a)s_{2:11}. \quad (A110)$$

From (A39), (A41), and (A42)

$$v(u_{2:1}) = \frac{2an^2\bar{w}^4\sigma_e^4}{\bar{q}^2}, \quad (A111)$$

$$v(u_{2:2}) = 2\sigma_e^4 \left[\frac{a\bar{w}^4}{\bar{q}^2} + N - a \right], \quad (A112)$$

and

$$\text{cov}(u_{2:1}, u_{2:2}) = \frac{2an\bar{w}^4\sigma_e^4}{\bar{q}^2}. \quad (A113)$$

Now substituting (A107) through (A113) into (A35), (A36), and (A37)

$$\begin{aligned}
 v(\hat{\sigma}_{2e}^2) &= \frac{2\sigma_e^4}{\Delta_2^2} \left[\left(\text{an}\bar{w}^2 \right)^2 \left(\frac{\text{an}^2\bar{w}^4}{\bar{q}^2} \right) + \left(\text{an}^2\bar{w}^2 \right)^2 \left(\frac{\text{a}\bar{w}^4}{\bar{q}^2} + N - a \right) \right. \\
 &\quad \left. - 2 \left(\text{an}^2\bar{w}^2 \right) \left(\text{an}\bar{w}^2 \right) \left(\frac{\text{an}\bar{w}^4}{\bar{q}^2} \right) \right] \\
 &= \frac{2\sigma_e^4 (N - a) (\text{an}^2\bar{w}^2)^2}{(N - a)^2 (\text{an}^2\bar{w}^2)^2} = \frac{2\sigma_e^4}{N - a}, \tag{A114}
 \end{aligned}$$

$$\begin{aligned}
 v(\hat{\sigma}_{2a}^2) &= \frac{2\sigma_e^4}{\Delta_2^2} \left[\left(\text{a}\bar{w}^2 + N - a \right)^2 \left(\frac{\text{an}^2\bar{w}^4}{\bar{q}^2} \right) + \left(\text{an}\bar{w}^2 \right)^2 \left(\frac{\text{a}\bar{w}^4}{\bar{q}^2} + N - a \right) \right. \\
 &\quad \left. - 2 \left(\text{an}\bar{w}^2 \right) \left(\text{a}\bar{w}^2 + N - a \right) \left(\frac{\text{an}\bar{w}^4}{\bar{q}^2} \right) \right] \\
 &= \frac{2\sigma_e^4 \left[(N - a)^2 \frac{\text{an}^2\bar{w}^4}{\bar{q}^2} + (N - a) \text{a}^2 \text{n}^2\bar{w}^4 \right]}{(N - a)^2 (\text{an}^2\bar{w}^2)^2} \\
 &= \frac{2\sigma_e^4}{\text{an}^2\bar{q}^2} + \frac{2\sigma_e^4}{n^2(N - a)} \\
 &= \frac{2(\sigma_e^2 + n\sigma_a^2)^2}{nN} + \frac{2\sigma_e^4}{n^2(N - a)}, \tag{A115}
 \end{aligned}$$

recalling that $\bar{q} = \frac{\sigma_e^2}{\sigma_e^2 + n\sigma_a^2}$ and $N = \text{an}$, and

$$\begin{aligned}
 \text{cov}(\hat{\sigma}_{2e}^2, \hat{\sigma}_{2a}^2) &= \frac{2\sigma_e^4}{\Delta_2^2} \left[- \left(\text{an}\bar{w}^2 \right) \left(\text{a}\bar{w}^2 + N - a \right) \left(\frac{\text{an}^2\bar{w}^4}{\bar{q}^2} \right) \right. \\
 &\quad \left. - \left(\text{an}^2\bar{w}^2 \right) \left(\text{an}\bar{w}^2 \right) \left(\frac{\text{a}\bar{w}^4}{\bar{q}^2} + N - a \right) \right. \\
 &\quad \left. + \left\{ \left(\text{an}^2\bar{w}^2 \right) \left(\text{a}\bar{w}^2 + N - a \right) + \left(\text{an}\bar{w}^2 \right)^2 \right\} \left(\frac{\text{an}\bar{w}^4}{\bar{q}^2} \right) \right]
 \end{aligned}$$

$$= \frac{2\sigma_e^4 [-(N-a)a^2 n^3 \bar{w}_i^4]}{(N-a)^2 (an^2 \bar{w}_i^2)^2}$$

$$= \frac{-2\sigma_e^4}{n(N-a)} \quad (116)$$

Note that (A114), (A115) and (A116) equal respectively (A104), (A105), and (A106) and are the variances and covariance of the usual AOV estimators when $\mu \equiv 0$.

3. MIVQUE with $\mu \neq 0$

Under balance (A43) gives $k_1 = k_2 = \dots = k_a = \bar{k}$, and $k = \frac{1}{a\bar{k}}$. Then from (A55) through (A59) we have

$$s_{3:11} = a\bar{k}^2 - 2\bar{k}^2 + \bar{k}^2 = (a-1)\bar{k}^2, \quad (A117)$$

$$s_{3:12} = \frac{a\bar{k}^2}{n} - \frac{2\bar{k}^2}{n} + \frac{\bar{k}^2}{n} = \frac{(a-1)\bar{k}^2}{n}, \quad (A118)$$

$$s_{3:22} = \frac{N-a}{\sigma_e^4} + \frac{a\bar{k}^2}{n^2} - \frac{2\bar{k}^2}{n^2} + \frac{\bar{k}^2}{n^2}$$

$$= \frac{N-a}{\sigma_e^4} + (a-1) \frac{\bar{k}^2}{n^2}, \quad (A119)$$

$$\Delta_3 = s_{3:11}s_{3:22} - s_{3:12}^2$$

$$= \frac{(N-a)s_{3:11}}{\sigma_e^4}, \quad (A120)$$

$$u_{3:1} = \bar{k}^2 \sum \left(\bar{y}_{i\cdot} - \frac{1}{a\bar{k}} \bar{k} \cdot \sum \bar{y}_{i\cdot} \right)^2$$

$$\begin{aligned}
 &= \bar{k}^2 \sum \left(\bar{y}_{i.} - \frac{a\bar{y}_{..}}{a} \right)^2 \\
 &= \bar{k}^2 \sum \left(\bar{y}_{i.} - \bar{y}_{..} \right)^2 \\
 &= \frac{\bar{k}^2 \text{SSA}_m}{n}, \tag{A121}
 \end{aligned}$$

and by similar steps

$$u_{3:2} = \frac{\text{SSE}}{\sigma_e^4} + \frac{\bar{k}^2 \text{SSA}_m}{n^2}, \tag{A122}$$

where SSE and SSA_m are respectively the usual error and treatment sums of squares and MSE and MSA_m are the corresponding mean squares for the 1-way AOV with mean μ .

Substituting (A117) through (A122) into (A60) and (A61) gets

$$\begin{aligned}
 \hat{\sigma}_{3e}^2 &= \frac{\sigma_e^4}{(N-a)s_{3:11}} \left[- \left(\frac{(a-1)\bar{k}^2}{n} \right) \left(\frac{\bar{k}^2 \text{SSA}_m}{n} \right) \right. \\
 &\quad \left. + \left((a-1)\bar{k}^2 \right) \left(\frac{\text{SSE}}{\sigma_e^4} + \frac{\bar{k}^2 \text{SSA}_m}{n^2} \right) \right] \\
 &= \frac{\sigma_e^4 (a-1)\bar{k}^2 \text{SSE}}{(N-a)\sigma_e^4 s_{3:11}} = \frac{\text{SSE}}{N-a} = \text{MSE} \tag{A123}
 \end{aligned}$$

and

$$\begin{aligned}
 \hat{\sigma}_{3a}^2 &= \frac{\sigma_e^4}{(N-a)s_{3:11}} \left[\left(\frac{N-a}{\sigma_e^4} + (a-1)\frac{\bar{k}^2}{n^2} \right) \left(\frac{\bar{k}^2 \text{SSA}_m}{n} \right) \right. \\
 &\quad \left. - \left(\frac{(a-1)\bar{k}^2}{n} \right) \left(\frac{\text{SSE}}{\sigma_e^4} + \frac{\bar{k}^2 \text{SSA}_m}{n^2} \right) \right]
 \end{aligned}$$

$$\begin{aligned}
 &= \frac{\sigma_e^4}{(N - a)(a - 1)\bar{k}^2} \left[\frac{(N - a)\bar{k}^2 \text{SSA}_m}{n\sigma_e^4} - \frac{(a - 1)\bar{k}^2 \text{SSE}}{n\sigma_e^4} \right] \\
 &= \frac{1}{n} \left[\frac{\text{SSA}_m}{a - 1} - \frac{\text{SSE}}{N - a} \right] \\
 &= \frac{1}{n} \left[\text{MSA}_m - \text{MSE} \right] . \tag{A124}
 \end{aligned}$$

Using (A117) through (A120) with (A68), (A69), and (A70) yields

$$v(\hat{\sigma}_{3e}^2) = \frac{2s_{3:11}}{\Delta_3} = \frac{2\sigma_e^4}{N - a} , \tag{A125}$$

$$\begin{aligned}
 v(\hat{\sigma}_{3a}^2) &= \frac{2s_{3:22}}{\Delta_3} \\
 &= \frac{2\sigma_e^4 \left[\frac{N - a}{\sigma_e^4} + (a - 1) \frac{\bar{k}^2}{n^2} \right]}{(N - a)(a - 1)\bar{k}^2} \\
 &= \frac{2}{n^2} \left[\frac{(\sigma_e^2 + n\sigma_a^2)^2}{a - 1} + \frac{\sigma_e^4}{N - a} \right] , \tag{A126}
 \end{aligned}$$

$$\begin{aligned}
 \text{and } \text{cov}(\hat{\sigma}_{3e}^2, \hat{\sigma}_{3a}^2) &= \frac{-2s_{3:12}}{\Delta_3} \\
 &= \frac{-2\sigma_e^4(a - 1)\bar{k}^2}{n(N - a)(a - 1)\bar{k}^2} \\
 &= \frac{-2\sigma_e^4}{n(N - a)} , \tag{A127}
 \end{aligned}$$

where (A125), (A126), and (A127) are the variances and covariance of the usual AOV estimators under balance.

4. MINQUE with $\mu \neq 0$

The results in (A123) and (A124) demonstrate that the MINQUE estimators, being a special case of the MIVQUE estimators, reduce to the usual ACV estimators under balance.

For $\ell_1 = \ell_2 = \dots = \ell_a = \bar{\ell}$ in (A71) and $\sigma_e^2 = \sigma_a^2 = 1$ in (A117) through (A120) we get

$$s_{4:11} = (a - 1)\bar{\ell}^2, \quad (A128)$$

$$s_{4:12} = \frac{(a - 1)\bar{\ell}^2}{n} = (a - 1)\bar{\ell}(1 - \bar{\ell}), \quad (A129)$$

$$\begin{aligned} s_{4:22} &= N - a + (a - 1)\frac{\bar{\ell}^2}{n^2} \\ &= N - a + (a - 1)(1 - \bar{\ell})^2, \end{aligned} \quad (A130)$$

and $\Delta_4 = (N - a)s_{4:11}, \quad (A131)$

using the identity $\frac{\bar{\ell}}{n} = 1 - \bar{\ell}$.

Under balance (A89), (A95), and (A93) become

$$\begin{aligned} v(u_{4:1}) &= 2 \left[a \frac{\bar{\ell}^4}{\bar{k}^2} + 2 \frac{\bar{\ell}^4}{\bar{k}^2} - 4 \frac{\bar{\ell}^4}{\bar{k}^2} + \frac{\bar{\ell}^4}{\bar{k}^2} + 2 \frac{\bar{\ell}^4}{\bar{k}^2} - 4 \frac{\bar{\ell}^4}{\bar{k}^2} + 2 \frac{\bar{\ell}^4}{\bar{k}^2} \right] \\ &= 2(a - 1)\frac{\bar{\ell}^4}{\bar{k}^2}, \end{aligned} \quad (A132)$$

$$\text{cov}(u_{4:1}, u_{4:2}) = 2 \left[-(a - 1)\frac{\bar{\ell}^4}{\bar{k}^2} + a \frac{\bar{\ell}^3}{\bar{k}^2} - 3 \frac{\bar{\ell}^3}{\bar{k}^2} + \frac{\bar{\ell}^3}{\bar{k}^2} - \frac{\bar{\ell}^3}{\bar{k}^2} + 2 \frac{\bar{\ell}^3}{\bar{k}^2} \right]$$

$$\begin{aligned}
 &= 2(a - 1) \left(\frac{\bar{\ell}_\cdot^3}{\bar{k}_\cdot^2} - \frac{\bar{\ell}_\cdot^4}{\bar{k}_\cdot^2} \right) \\
 &= 2(a - 1) \frac{\bar{\ell}_\cdot^3}{\bar{k}_\cdot^2} (1 - \bar{\ell}_\cdot) , \tag{A133}
 \end{aligned}$$

and

$$\begin{aligned}
 v(u_{4:2}) &= 2 \left[-(a - 1) \frac{\bar{\ell}_\cdot^4}{\bar{k}_\cdot^2} - 2(a - 1) \left(\frac{\bar{\ell}_\cdot^3}{\bar{k}_\cdot^2} - \frac{\bar{\ell}_\cdot^4}{\bar{k}_\cdot^2} \right) + N\sigma_e^4 \right. \\
 &\quad \left. + 2N\sigma_e^2\sigma_a^2 + nN\sigma_a^4 - a\frac{n^2}{\bar{k}_\cdot^2} + a\frac{\bar{\ell}_\cdot^2}{\bar{k}_\cdot^2} - 2\frac{\bar{\ell}_\cdot^2}{\bar{k}_\cdot^2} + \frac{\bar{\ell}_\cdot^2}{\bar{k}_\cdot^2} \right] \\
 &= 2 \left[(a - 1) \left(\frac{\bar{\ell}_\cdot^2}{\bar{k}_\cdot^2} - 2\frac{\bar{\ell}_\cdot^3}{\bar{k}_\cdot^2} + \frac{\bar{\ell}_\cdot^4}{\bar{k}_\cdot^2} \right) + N\sigma_e^4 + 2N\sigma_e^2\sigma_a^2 + nN\sigma_a^4 \right. \\
 &\quad \left. - a(\sigma_e^2 + n\sigma_a^2)^2 \right] \\
 &= 2 \left[(a - 1) \frac{\bar{\ell}_\cdot^2}{\bar{k}_\cdot^2} (1 - \bar{\ell}_\cdot)^2 + (N - a)\sigma_e^4 \right] , \tag{A134}
 \end{aligned}$$

remembering that $\bar{k}_\cdot = \frac{n}{\sigma_e^2 + n\sigma_a^2}$.

Then substituting (A128) through (A134) into (A79), (A80) and (A81) we get

$$\begin{aligned}
 v(\hat{\sigma}_{4e}^2) &= \frac{2}{\Delta_4^2} \left[\left\{ (a - 1)\bar{\ell}_\cdot(1 - \bar{\ell}_\cdot) \right\}^2 \left\{ (a - 1) \frac{\bar{\ell}_\cdot^4}{\bar{k}_\cdot^2} \right\} \right. \\
 &\quad \left. + \left\{ (a - 1)\bar{\ell}_\cdot^2 \right\}^2 \left\{ (a - 1) \frac{\bar{\ell}_\cdot^2}{\bar{k}_\cdot^2} (1 - \bar{\ell}_\cdot)^2 + (N - a)\sigma_e^4 \right\} \right. \\
 &\quad \left. - 2 \left\{ (a - 1)\bar{\ell}_\cdot^2 \right\} \left\{ (a - 1)\bar{\ell}_\cdot(1 - \bar{\ell}_\cdot) \right\} \left\{ (a - 1) \frac{\bar{\ell}_\cdot^3}{\bar{k}_\cdot^2} (1 - \bar{\ell}_\cdot) \right\} \right]
 \end{aligned}$$

$$= \frac{2(a-1)^2(N-a)\bar{\ell}_e^4\sigma_e^4}{(N-a)^2(a-1)^2\bar{\ell}_e^4} = \frac{2\sigma_e^4}{N-a} \quad (A135)$$

$$\begin{aligned} v(\hat{\sigma}_{4a}^2) &= \frac{2}{n^2} \left[\left\{ N-a+(a-1)(1-\bar{\ell}_e)^2 \right\}^2 \left\{ (a-1) \frac{\bar{\ell}_e^4}{k^2} \right. \right. \\ &\quad + \left\{ (a-1)\bar{\ell}_e(1-\bar{\ell}_e) \right\}^2 \left\{ (a-1) \frac{\bar{\ell}_e^2}{k^2} (1-\bar{\ell}_e)^2 + (N-a)\sigma_e^4 \right\} \\ &\quad - 2 \left\{ (a-1)\bar{\ell}_e(1-\bar{\ell}_e) \right\} \left\{ N-a+(a-1)(1-\bar{\ell}_e)^2 \right\} \\ &\quad \left. \times \left\{ (a-1) \frac{\bar{\ell}_e^3}{k^2} (1-\bar{\ell}_e) \right\} \right] \\ &= \frac{2 \left[(N-a)^2(a-1) \frac{\bar{\ell}_e^4}{k^2} + (N-a)(a-1)^2 \bar{\ell}_e^2 (1-\bar{\ell}_e)^2 \sigma_e^4 \right]}{(N-a)^2(a-1)^2 \bar{\ell}_e^4} \\ &= \frac{2}{(a-1)k^2} + \frac{2\sigma_e^4}{n^2(N-a)} \\ &= \frac{2}{n^2} \left[\frac{(\sigma_e^2 + n\sigma_a^2)^2}{a-1} + \frac{\sigma_e^4}{N-a} \right], \quad (A136) \end{aligned}$$

$$\begin{aligned} \text{and } \text{cov}(\hat{\sigma}_{4e}^2, \hat{\sigma}_{4a}^2) &= \frac{2}{n^2} \left[- \left\{ (a-1)\bar{\ell}_e(1-\bar{\ell}_e) \right\} \left\{ N-a+(a-1)(1-\bar{\ell}_e)^2 \right\} \left\{ (a-1) \frac{\bar{\ell}_e^4}{k^2} \right. \right. \\ &\quad - \left\{ (a-1)\bar{\ell}_e^2 \right\} \left\{ (a-1)\bar{\ell}_e(1-\bar{\ell}_e) \right\} \left\{ (a-1) \frac{\bar{\ell}_e^2}{k^2} (1-\bar{\ell}_e)^2 + (N-a)\sigma_e^4 \right\} \\ &\quad \left. + \left\{ (a-1)\bar{\ell}_e^2 \right\} \left\{ N-a+(a-1)(1-\bar{\ell}_e)^2 \right\} + \left\{ (a-1)\bar{\ell}_e(1-\bar{\ell}_e) \right\}^2 \right] \\ &\quad \times \left\{ (a-1) \frac{\bar{\ell}_e^3}{k^2} (1-\bar{\ell}_e) \right\} \end{aligned}$$

$$\begin{aligned}
 &= \frac{2[-(N - a)(a - 1)^2 \bar{l}^3 (1 - \bar{l}) \sigma_e^4]}{(N - a)^2 (a - 1)^2 \bar{l}^4} \\
 &= \frac{-2\sigma_e^4}{n(N - a)} \quad . \quad (A137)
 \end{aligned}$$

Equations (A135), (A136), and (A137) equal respectively (A125), (A126), and (A127) and are the variances and covariance under balance of the usual AOV estimators.

REFERENCES

- Swallow, W. H. and S. R. Searle [1973]. Minimum norm and minimum variance quadratic unbiased estimators of variance components from unbalanced data in the 1-way classification. Paper No. BU-447-M in the Mimeograph Series, Biometrics Unit, Cornell University, Ithaca, New York.
- Townsend, E. C. and S. R. Searle [1971]. Best quadratic unbiased estimation of variance components from unbalanced data in the 1-way classification. Biometrics 27, 643-657.